Differentiation

if $y = f(x)$ ... be a given function.

$$y = f(x)$$

Infinitiely small increasment on both sides by δy in x and y; we get

$$y + \delta y = f(x + \delta x)$$
$$\delta y = f(x + \delta x) - y$$
$$\delta y = f(x + \delta x) - f(x).$$

On dividing on the both side w.r.t δx

$$\frac{\lim}{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{\lim}{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$\frac{dy}{dx} = \frac{\lim}{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

So,

$\frac{df(x)}{dx}$ is read as differentiation coefficient of $f(x)$ w.r.t x.

Differentation coefficient of $f(x)$ w.r.t x.

Definiton → The limiting value of ratio of Infinitely small increasement in y and x. is called differentation.

it is also called 'derivative' and is defined as $\frac{\lim}{x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx} = \frac{df(x)}{dx} = \frac{\lim}{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$

$f'(x) = y_1, \; y' = y^\circ = D_1 = D' = D(y)$

I. Derivatives of some function

$\frac{dx^n}{dx} = nx^{n-1}$ $\qquad$ $\frac{d\sec x}{dx} = \sec x \tan x$

$\frac{d\sin x}{dx} = \cos x$ $\qquad$ $\frac{d\,\mathrm{cosec}\,x}{dx} = -\mathrm{cosec}\,x \cot x$

$\frac{d\cos x}{dx} = -\sin x$ $\qquad$ $\frac{d\log x}{dx} = \frac{1}{x}$ $\qquad$ $\frac{dc}{dx} = 0$ $\quad c$ = constant

$\frac{d\tan x}{dx} = \sec^2 x$ $\qquad$ $\frac{de^x}{dx} = e^x$

$\frac{d\cot x}{dx} = -\mathrm{cosec}^2 x$ $\qquad$ $\frac{da^x}{dx} = a^x \log_e a$

Some Rule for differentiation –

$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

$\frac{d(u-v)}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

$\frac{d(u \cdot v)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{vu' - uv'}{v^2}$ $\qquad \left[\because u' = \frac{du}{dx},\ v' = \frac{dv}{dx}\right]$

CHAIN RULE –

$y = f(t)$ and $t = g(x)$

$\frac{dy}{dx} = \frac{df(t)}{dx} = \frac{df\{g(x)\}}{dg(x)} \cdot \frac{dg(x)}{dx}$

2. Derivatives of Exponential and logarithm function.

Logarithmic Rule –

$\log(xy) = \log(x) + \log y$ $\quad \log x^n = n\log x$

$\log\left(\frac{x}{y}\right) = \log x - \log y$ $\quad \log_a x = \frac{1}{\log_x a}$

$\log_x x = 1$ & $\log 1 = 0$

if $a^x = b$; then $\log_a b = x$

$\frac{de^x}{dx} = e^x$ $\quad \frac{d\log x}{dx} = \frac{1}{x}$

$\frac{da^x}{dx} = a^x \log a$ $\quad \frac{d(\log_a x)}{dx} = \frac{1}{x\log a}$.

3. Derivatives of Inverse trigonometric Function

$\frac{d\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}$ $\quad \frac{d\cot^{-1}x}{dx} = \frac{-1}{1+x^2}$

$\frac{d\cos^{-1}x}{dx} = \frac{-1}{\sqrt{1-x^2}}$ $\quad \frac{d\sec^{-1}x}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

$\frac{d\tan^{-1}x}{dx} = \frac{1}{1+x^2}$ $\quad \frac{d\operatorname{cosec}^{-1}x}{dx} = \frac{-1}{|x|\sqrt{x^2-1}}$

4. Differentiation by trigonometrical Transformation

$1 - \cos x = 2\sin^2 x/2$

$1 + \cos x = 2\cos^2 x/2$

$\sin 3x = 3\sin x - 4\sin^3 x$

$\cos 3x = 4\cos^3 x - 3\cos x$

$\sin x = \sqrt{1 - \cos^2 x}$

$\cos x = \sqrt{1 - \sin^2 x}$

$\sin 2x = \frac{2\tan x}{1 + \tan^2 x}$; $\sin x = \frac{2\tan x/2}{1 + \tan^2 x/2}$

$\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$; $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$; $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$; $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

Some useful subsitrafionx.

$f(x) = \sqrt{a^2 - x^2}$, put $x = a\sin\theta$, $x = a\cos\theta$

$f(x) = \sqrt{a^2 + x^2}$, put $x = a\tan\theta$, $x = a\cot\theta$

$f(x) = \sqrt{x^2 - a^2}$, put $x = a\sec\theta$, $x = a\operatorname{cosec}\theta$

$f(x) = \sqrt{a - x}$, put $x = a\cos 2\theta$.

Differentiation of implicit functions

Ex:- $x^3 + y^3 = 3axy$; find $\frac{dy}{dx}$

$$\frac{dx^3}{dx} + \frac{dy^3}{dx} = \frac{d(3axy)}{dx}$$

$$3x^2 + 3y^2\frac{dy}{dx} + 3a\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3ax\frac{dy}{dx} + 3ay$$

$$3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay - 3x^2$$

$$\frac{dy}{dx}(3y^2 - 3ax) = 3ay - 3x^2$$

$$\frac{dy}{dx} = \frac{3ay - 3x^2}{3y^2 - 3ax}$$

Ex:- $y = \frac{1}{(1+\tan 3x)^2}$; Find $\frac{dy}{dx}$

Ex:- $y = \log xy + y^3$, find $\frac{dy}{dx}$

Ex:- $x^2 - y^2 + xy + 7 = \tan x + \sin y$; find dy/dx

Ex:- $y = \frac{1-\tan^2 x}{1-\cos x}$; find dy/dx

Ex:- $x^2 + xy + x^2y + 5 = \pi$

Differentiation using logarithm

दिए गए function के पावर में कुछ expression or उसके multiply में expression हो, तो दोनों तरफ logarithmic का use करके ---- Differentiate करेंगे।

$y = f(x)^{f(x)}$ or $f(x)^{g(x)}$ or $f(x)^{g(x)^{h(x)}}$ or $f(x)g(x)h(x)$.

Ex: $\log_2 8 = 3 \Rightarrow 2^3 = 8$

$\log_3 81 = 4 \Rightarrow 3^4 = 81$

$\log_5 125 = 3 \Rightarrow 5^3 = 125$

$\log_e a = x \Rightarrow e^x = a$

let $y = a^x$; then

Taking log on the both side

$\log y = \log a^x$

$\log y = x \log a$

$\log_e y = x \log a$

$y = e^{x \log a}$

$$\boxed{a^x = y = e^{x \log a}}$$

Ex: $x^x = e^{x\log x}$

$m^n = e^{n\log m}$

$\sin x^{\tan x} = e^{\tan x \log \sin x}$

$x^{\tan x} = e^{\tan x \log x}$

$\sin x^x = e^{x \log \sin x}$

$$y = e^x \cos^3 x \sin^2 x$$

Taking log on the both side

$$\log y = \log(e^x \cos^3 x \sin^2 x)$$

$$\log y = \log e^x + \log \cos^3 x + \log \sin^2 x$$

Differentiate w.r.t to x

$$\frac{\log y}{dy} \cdot \frac{dy}{dx} = \frac{d \log e^x}{de^x} \cdot \frac{de^x}{dx} + 3\log \cos x \frac{d \cos x}{dx} + \frac{d \log \sin x}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = 1 - 3\tan x + 2\cot x$$

$$\frac{dy}{dx} = y(1 - 3\tan x + 2\cot x)$$

$$\boxed{\frac{dy}{dx} = e^x \cos^3 x \sin^2 x (1 - 3\tan x + 2\cot x)}$$

Ex: $y = x^x$: find dy/dx.

$$y = x^x = e^{x \log x}$$

$$\frac{dy}{dx} = \frac{de^{x\log x}}{dx \log x} \cdot \frac{dx \log x}{dx}$$

$$= e^{x \log x}(1 + \log x)$$

$$= x^x (1 + \log x).$$

Derivatives of an infinite series :-

if we take out a single term form an infinite series, it remains unaffected. we utilize this result in finding the derivative.

$y = x^{x^{x^{x} \cdots \text{to } \infty}}$ $\quad \frac{dy}{dx} = \frac{y^2}{x(1-y\log x)}$

$y = x^y$

$\log y = y \log x$

$\frac{d\log y}{dy} \cdot \frac{dy}{dx} = y \frac{d\log x}{dx} + \log x \frac{dy}{dx}$

$\frac{1}{y}\frac{dy}{dx} = \frac{y}{x} + \log x \frac{dy}{dx}$

$\frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x}$

$\frac{dy}{dx} = \frac{y^2}{x(1-\log x)}$

$y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots \text{to } \infty}}}$

$y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots \infty}}}$

$y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \cdots \infty}}}$

$y = \sqrt{\cos x + \sqrt{\cos x + \cos x + \cdots + \text{to } \infty}}$

$y = a^{x^{a^{x^{a^{x} \cdots \text{to } \infty}}}}$

$y = \cos x^{\cos x^{\cos x \cdots \text{to } \infty}}$.

8. Derivative of one function w.r.t another function.

$f(x)$ & $g(x)$ are two functions.

$u = f(x)$; $v = g(x)$: Find $\frac{dy}{dv} = \frac{du/dx}{dv/dx}$

Ex:- e^x w.r.t $\sqrt{x}$

$$\frac{de^x}{d\sqrt{x}} = \frac{de^x/dx}{d\sqrt{x}/dx} = \frac{e^x}{1/2\sqrt{x}} = 2\sqrt{x}e^x.$$

Ex. $\sin^2 x$ w.r.t $e^{\cos x}$

$$\frac{d\sin^2 x}{de^{\cos x}} = \frac{d\sin^2 x/dx}{de^{\cos x}/dx} = \frac{2\sin x \cos x}{e^{\cos x}(-\sin x)} = \frac{-2\cos x \sin x}{e^{\cos x}}$$

$$= -$$

9. Derivatives of parametric functions.

$x = f(t)$ and $y = g(t)$

$$\frac{dx}{dt} = \frac{df(t)}{dt} \quad ; \quad \frac{dy}{dt} = \frac{dg(t)}{dt}$$

$x = a(t + \sin t)$ & $y = a(1 - \cos t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin t/2 \cos t/2}{2\cos^2 t/2} = \tan$$

Second - order derivatives -

$y' = \frac{dy}{dx}$ ⟶ first order derivatives.

$\frac{dy'}{dx} = y^2$ ⟶ second order derivatives.

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

y_2 , $\ddot{y}$, $f''(x)$, $\dot{y}''$, D_2 , D'' , $D^2(y)$.

Ex: $y = x^{19}$

$y' = 19x^{18}$

$y'' = 19 \times 18 x^{17}$. ∎