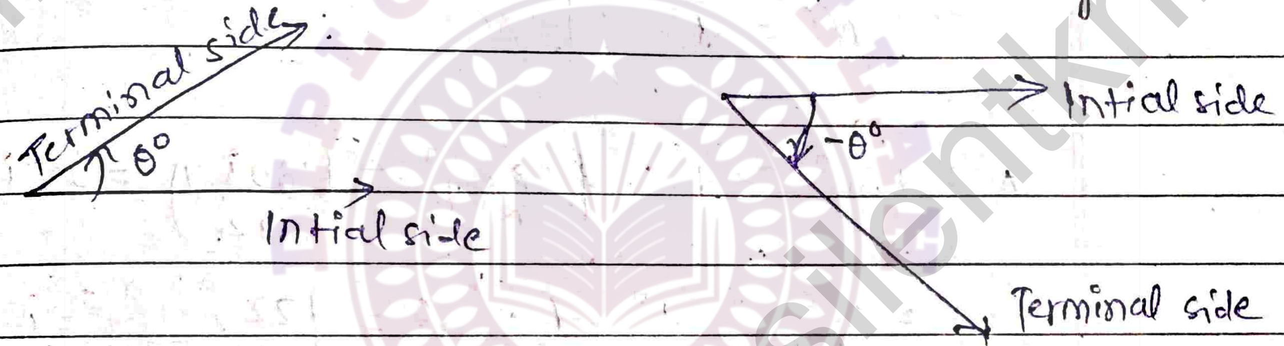
Welcome to 2nd semester

UNIT - I Trigonometry

MATHEMATICS

Trigonometry - The branch of mathematics which deals with the measurements of angles of triangles and the problems related to these angles.

Measure of angle → The amount of rotation from the Intial side to the terminal side is called Measure of the angle.



- **Sexagesimal system (Degree Measure)** - The angle traced by a moving line about a point from its intial position to the terminating position in making $\frac{1}{360}$ of the complete revolution of a circle. It is said to be $1°$ (1 degree).

$$1' = \frac{1°}{60'}$$

$$1'' = \frac{1'}{60'}$$

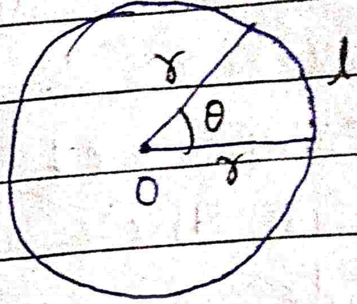
$$\boxed{1 \text{ right angle} = 90°}$$

$$\boxed{1° = 60'} \text{ and } \boxed{1' = 60''}$$

(ii) Circular system (Radian measure) —

A radian is an angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.

$$\boxed{\theta = \frac{l}{r}}$$



$$\boxed{\pi^c = 180^\circ}$$

$$\pi^c = 180^\circ \Rightarrow 1^c = \left(\frac{180}{\pi}\right)^\circ = \left(\frac{180 \times 7}{22}\right)^\circ = 57^\circ 16' 21''$$

$$180^\circ = \pi^c = 1^\circ = \left(\frac{\pi}{180}\right)^c = \left(\frac{22}{7} \times \frac{1}{180}\right)^c = 0.01746^c$$

(i) In the Degree measure, we measure angles in degree, minutes & seconds.

1 right angle $= 90^\circ$

$1^\circ = 60'$

$1' = 60''$

(ii) In the circular measure, we measure angles in radians. $\pi^c = 180^\circ$

(iii) if an arc of length l makes an angle θ^c at the centre of a circle of radius r.

$$\theta = \frac{l}{r}.$$

Q. $\left(\frac{7\pi}{12}\right) =$ $1^c = \left(\frac{180}{\pi}\right)^\circ$

$\left(\frac{7\pi}{12}\right)^c = \left(\frac{\cancel{180}^{15}}{\cancel{\pi}} \times \frac{7\cancel{\pi}}{\cancel{12}}\right)^\circ = 105^\circ.$

Q. $15^\circ =$ $1^\circ = \left(\frac{\pi}{180}\right)^c$

$= \left(\frac{\pi}{180} \times \cancel{15}\right)^c = \left(\frac{22}{7} \times \frac{1}{180} \times 15\right)^c = \left(\frac{\pi}{12}\right)^c.$

$=$

$\left(\frac{3}{4}\right)^c$ $1^c = \left(\frac{180}{\pi}\right)^\circ$

$\frac{3}{4}^c = \left(\frac{180}{\pi} \times \frac{3}{4}\right)^\circ = \left(\frac{180 \times 7}{22} \times \frac{3}{4}\right)^\circ$

$= 42^\circ 57' 16''.$

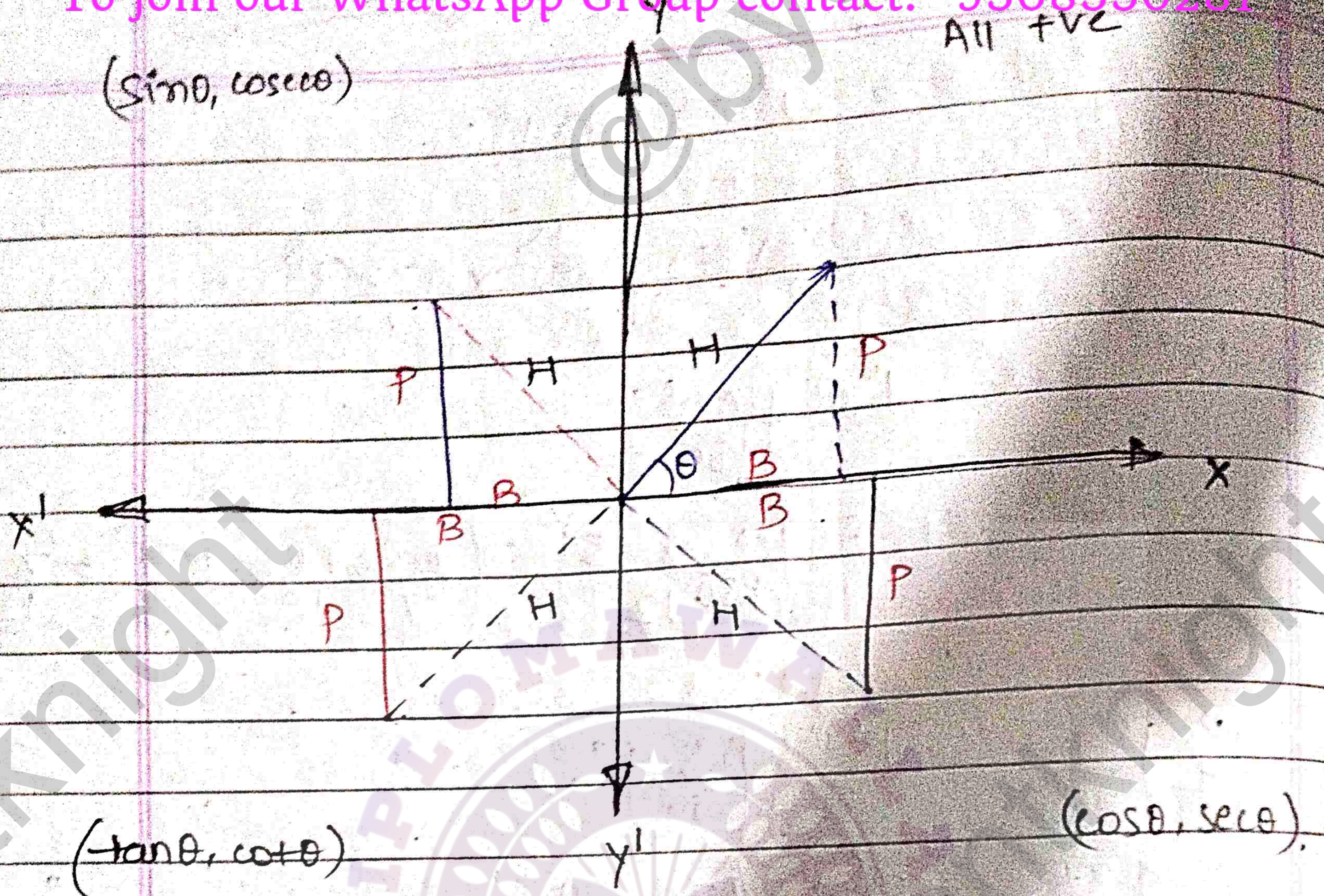
$-37^\circ 30' \Rightarrow 1^\circ = \left(\frac{\pi}{180}\right)^c$

$-37^\circ 30' = \left\{\frac{\pi}{180} \times -37^\circ 30'\right\}^c$

$= \frac{\pi}{180} \times \left(\frac{37\frac{1}{2}}{2}\right)^\circ$

$= \frac{-\pi}{180} \times \frac{75}{2}^\circ = \left(\frac{-5\pi}{24}\right)^c$

Hence, $(-37^\circ 30') = \left(\frac{-5\pi}{24}\right)^c.$



$\sin\theta = \frac{P}{H}$, $\operatorname{cosec}\theta = \frac{H}{P}$

$\cos\theta = \frac{B}{H}$, $\sec\theta = \frac{H}{B}$

$\tan\theta = \frac{P}{B}$, $\cot\theta = \frac{B}{P}$

$\sec\theta = \frac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, $\tan\theta = \frac{1}{\cot\theta}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \cot^2\theta = \operatorname{cosec}^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$

Negative [illegible] length.

$\sin(-\theta) = -\sin\theta$, $\tan(-\theta) = -\tan\theta$

$\cos(-\theta) = \cos\theta$, $\sec(-\theta) = \sec\theta$

$\sec(-\theta) = \sec\theta$, $\text{cosec}(-\theta) = -\text{cosec}\theta$.

$-\theta$

- $\sin(2n\pi+\theta) = \sin\theta$, $\text{cosec}(2\pi n+\theta) = \text{cosec}\theta$
- $\cos(2n\pi+\theta) = \cos\theta$, $\sec(2n\pi+\theta) = \sec\theta$
- $\tan(n\pi+\theta) = \tan\theta$, $\cot(n\pi+\theta) = \cot\theta$

①

- $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$, $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$.
- $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$, $\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$.
- $\sec\left(\frac{\pi}{2}-\theta\right) = \text{cosec}\theta$, $\text{cosec}\left(\frac{\pi}{2}-\theta\right) = \sec\theta$

②

$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$, $\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$

$\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$, $\cot\left(\frac{\pi}{2}+\theta\right) = -\tan\theta$

$\sec\left(\frac{\pi}{2}+\theta\right) = -\text{cosec}\theta$, $\text{cosec}\left(\frac{\pi}{2}+\theta\right) = \sec\theta$

③

$\cos(\pi-\theta) = -\cos\theta$

$\sin(\pi-\theta) = \sin\theta$

$\tan(\pi-\theta) = -\tan\theta$

④

$\cos(\pi+\theta) = -\cos\theta$

$\sin(\pi+\theta) = -\sin\theta$

$\tan(\pi+\theta) = \tan\theta$

(5)

$\cos(2\pi-\theta)=\cos\theta$
$\sin(2\pi-\theta)=-\sin\theta$
$\tan(2\pi-\theta)=-\tan\theta$

(6)

$\cos(2\pi+\theta)=\cos\theta$
$\sin(2\pi+\theta)=\sin\theta$
$\tan(2\pi+\theta)=\tan\theta$

(7)

$\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta$

$\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta$

(8)

$\cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta$

$\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta$

यदि Trigonometry function मे angle $(90-\theta)$, $(90+\theta)$, $(270-\theta)$, $(270+\theta)$ हो, तो function change करेगा, अथवा $(180-\theta)$, $(180+\theta)$, $(360-\theta)$, $(360+\theta)$ हो, तो function change नहीं करेगा।

* Trigonometric functions of Sum and difference of numbers.

1. $\cos(A+B)=\cos A\cos B-\sin A\sin B$
   $\cos(A-B)=\cos A\cos B+\sin A\sin B$

2. $\sin(A+B)=\sin A\cos B+\cos A\sin B$
   $\sin(A-B)=\sin A\cos B-\cos A\sin B$

3. $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\cdot\tan B}$, $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\cdot\tan B}$

4. $$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

$$\cot(A-B) = \frac{\cot A \cdot \cot B + 1}{\cot B - \cot A}$$

5. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

Some more important functions—

- $2\sin A \cos B = \sin(A+B) + \sin(A-B)$.
- $2\cos A \sin B = \sin(A+B) - \sin(A-B)$.
- $2\cos A \cos B = \cos(A+B) + \cos(A-B)$.
- $2\sin A \sin B = \cos(A+B) - \cos(A+B)$.

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Trigonometric function of multiples of 2

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4} = \cos 72^\circ$$

$$\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72^\circ$$

$$\cos 36^\circ = \frac{\sqrt{5}+1}{4} = \sin 54^\circ$$

$$\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54^\circ$$

Trigonometric functions of half angle.

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, $1-\cos\theta = 2\sin^2\frac{\theta}{2}$

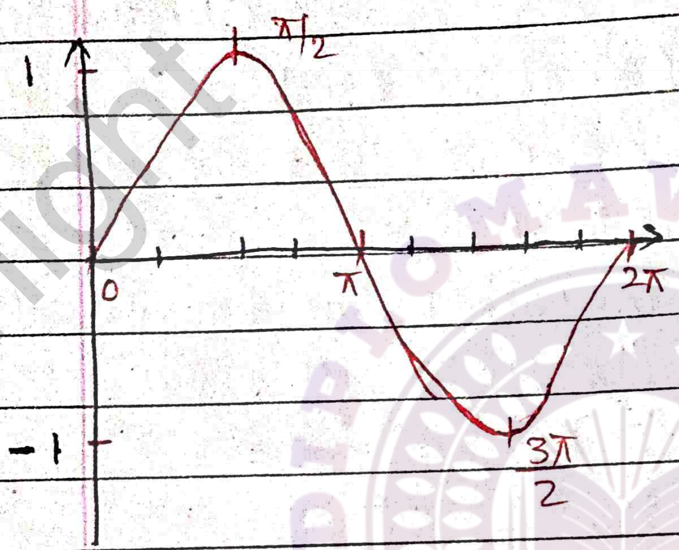
$1+\cos\theta = 2\cos^2\frac{\theta}{2}$, $\sin\theta = \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$

$$\cos\theta = \frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$$

Graph

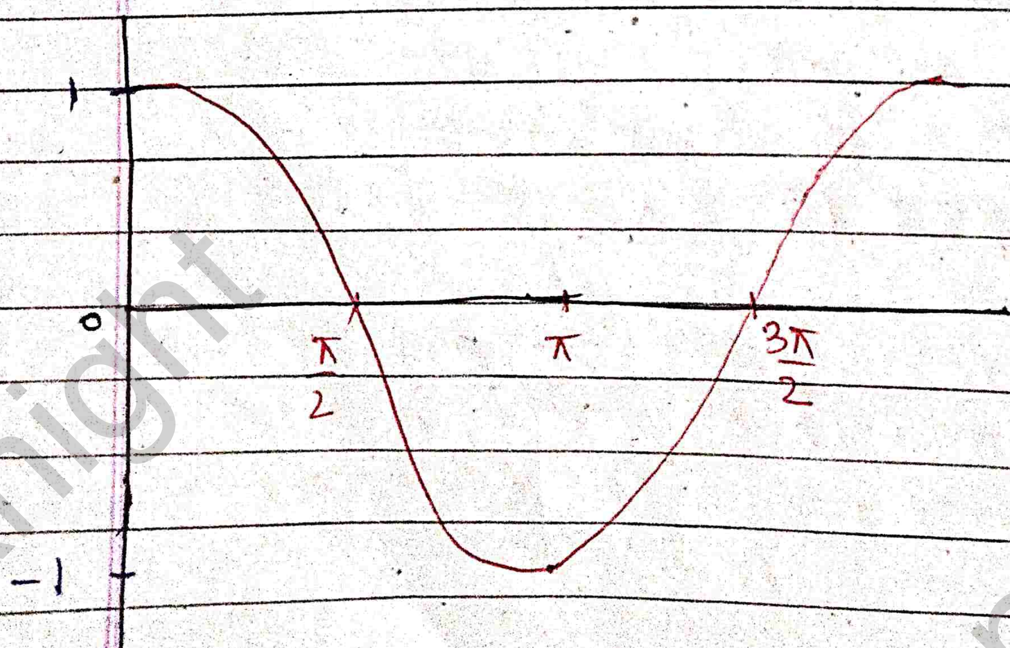
$y = \sin x$

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$3\pi/2$	2π
$\sin x$	0	$1/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/2$	0	$-1/2$	-1	0



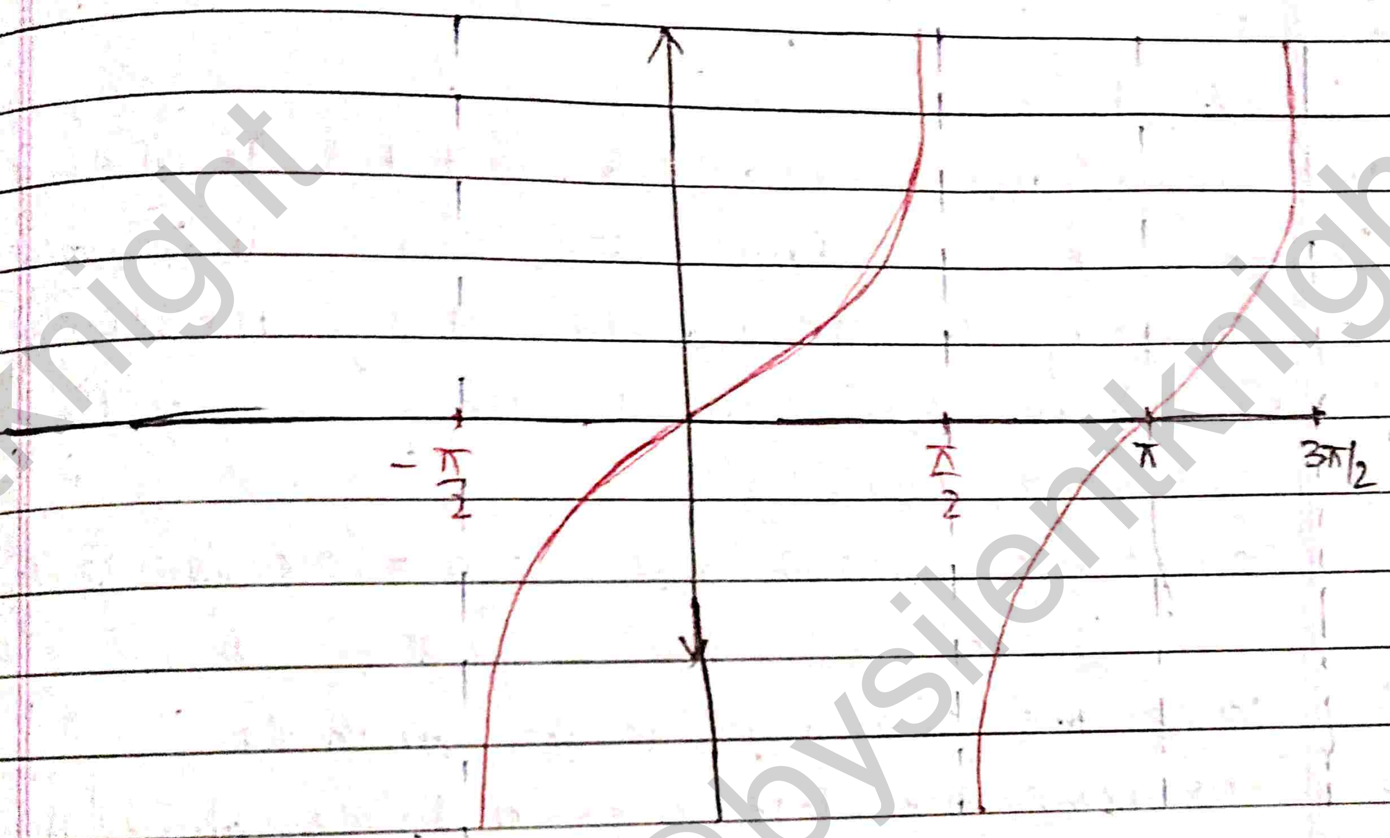
$y = \cos x$

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$3\pi/2$	2π
$\cos x$	1	$\sqrt{3}/2$	$1/2$	0	$-1/2$	$-\sqrt{3}/2$	-1	$-\sqrt{3}/2$	0	1



$y = \tan x$

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\tan x$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-1/\sqrt{3}$	0



Conditional Identities Involving the angles of a triangle

TYPE 1 Identities Involving sines and cosines.

Method–

1. Express the sum of the first two term as a product
2. In this product, Express the sum of two angles in terms of the third angle, using $A+B+C=\pi$
3. Expand the third term by using one of the following relations:
   $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = (2\cos^2\theta - 1), = (1-2\sin^2\theta)$
4. Take the common factor outside
5. Express the T-ratio of a single angle into a sum of two angles, and use the neccessary formula from the ones given in the box.

Q. if $A+B+C=\pi$, prove that $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$.

sol^n:– L.H.S $(\sin 2A + \sin 2B) + \sin 2C$

$$2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) + \sin 2C$$

$$2\sin(A+B)\cos(A-B) + 2\sin C\cos C$$

$$2\sin(\pi - C)\cos(A-B) + 2\sin C\cos C$$

$$2\sin C\cos(A-B) + 2\sin C\cos C.$$

$$2\sin C\left[\cos(A-B) + \cos C\right]$$

$$2\sin C\left[\cos(A+B) + \cos[\pi-(A+B)]\right]$$

$\left[\because A+B+C=\pi \quad C=\pi-(A+B)\right.$

$2\sin C\left[\cos(A+B)-\cos(A+B)\right]$

$2\sin C\left[2\sin\left(\frac{A+B-A+B}{2}\right)\sin\left(\frac{A+B+A-B}{2}\right)\right]$

$2\sin C\cdot 2\sin B\cdot\sin A$

$= 4\sin A\sin B\sin C.$

Q. If $A+B+C=\pi$ prove that $\cos 2A+\cos 2B+\cos 2C = -1-4\cos A\cos B\cos C.$

Type 2 Identities involving squares of sines and cosines

Method -

change the squares of sines and cosines into cosines of double the angles by using the formula.

$\cos^2\theta = \frac{(1+\cos 2\theta)}{2}$; $\sin^2\theta = \frac{(1-\cos 2\theta)}{2}$.

Q. If $A+B+C=\pi$, prove that $\sin^2 A+\sin^2 B+\sin^2 C = 2(1+\cos A\cos B\cos C).$

Sol $\sin^2 A+\sin^2 B+\sin^2 C$

$\frac{1-\cos 2A}{2}+\frac{1-\cos 2B}{2}+\frac{1-\cos 2C}{2}$

$= \frac{1-\cos 2A+1-\cos 2B+1-\cos 2C}{2}$

$= \frac{3-(\cos 2A+\cos 2B+\cos 2C)}{2}$

$$\frac{3}{2} - \frac{1}{2}(\cos 2A + \cos 2B + \cos 2C)$$

$$\frac{3}{2} - \frac{1}{2}\left\{2\cos\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) + \cos 2C\right\}$$

$$\frac{3}{2} - \frac{1}{2}\left\{2\cos(A+B)\cos(A-B) + \cos 2C\right\}$$

$$\frac{3}{2} - \frac{1}{2}\left\{2\cos(\pi - C)\cos(A-B) + 2\cos^2 C - 1\right\}$$

$$\frac{3}{2} - \frac{1}{2}\left(-2\cos C\cos(A-B) + 2\cos^2 C - 1\right)$$

$$\frac{3}{2} - \frac{1}{2}\left[-2\cos C\left\{\cos(A-B) - \cos C\right\}\right] + \frac{1}{2}$$

$$\frac{3}{2} - \frac{1}{2}\left[-2\cos C\left\{\cos(A-B) + \cos(\pi - (A+B))\right\}\right] + \frac{1}{2}$$

$$\frac{3}{2} - \frac{1}{2}\left[-2\cos C\left\{\cos(A-B) + \cos(A+B)\right\}\right] + \frac{1}{2}$$

$$\frac{3}{2} - \frac{1}{2}\left[-2\cos C\left\{2\cos A\cos B\right\}\right] + \frac{1}{2}$$

$$2 - \frac{1}{\cancel{2}} \times -\cancel{4}\cos A\cos B\cos C$$

$$2 + 2\cos A\cos B\cos C$$

$$2(1 + \cos A\cos B\cos C)$$ proved.

Type 3. Identities Involving tangents —

Method —

1. Using $A+B+C=\pi$, Express the sum of two angles in term of the third.

2. Take tangents on both sides and Expand LHS.

3. cross multiply and transpose.

Q. if $A+B+C=\pi$, P.T $\tan A+\tan B+\tan C=\tan A\cdot\tan B\cdot\tan C$.

Sol^n —
$$A+B=\pi-C$$
$$\tan(A+B)=\tan(\pi-C)$$
$$\frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$$
$$\tan A+\tan B=-\tan C+\tan A\cdot\tan B\cdot\tan C$$
$$\tan A+\tan B+\tan C=\tan A\cdot\tan B\tan C.$$