Matrices

Definition :-

A set of $(m \times n)$ numbers arranged in the form of an ordered set of 'm rows' and 'n columns' is called $m \times n$ matrix.

* $A = [a_{ij}]_{m \times n}$

where a_{ij} represents the element at the intersection of i^{th} row and j^{th} column.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Type of matrices

1. Square matrix :- A matrix in which the number of rows is equal to the number of column.

Example :

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 5 & -7 \\ 4 & 0 & 0 \end{bmatrix}_{3\times 3}$$

2. Horizontal matrix :- matrix in which number of columns are more than number of rows.

Example : $\begin{bmatrix} 2 & 5 & 7 \\ 1 & -1 & 0 \end{bmatrix}$

3. Vertical matrix :- Matrix in which number of rows are more than number of columns.

Example : $\begin{bmatrix} 2 & 1 \\ 5 & -1 \\ 7 & 0 \end{bmatrix}$

4. Row matrix :- A matrix having only one row is called a row matrix.

Example : $\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$

5. column matrix :- A matrix having only one column is called a column matrix.

Example $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

6. Diagonal matrix :- A square matrix $A = [a_{ij}]_{n \times n}$ is called a Diagonal matrix if $a_{ij} = 0$ for all $i \neq j$

A diagonal matrix of order $n \times n$ having $d_1, d_2, \ldots d_n$ as diagonal elements is denoted by $\text{diag}[d_1, d_2, \ldots d_n]$

Example : $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

7. Scalar matrix :- A diagonal matrix in which all the diagonal elements are equal.

Example : $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

8. **Identity or Unit matrix:** A diagonal matrix in which all diagonal elements are equal to one.

The Identity matrix of order n is denoted by I_n

Example: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$ $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$

9. **Null matrix:** A matrix whose all elements are zero is called a null matrix or a zero matrix.

Example: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

10. **Upper Triangular matrix:** A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if $a_{ij} = 0$ for all $i > j$

Example: $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix}$

11. **Lower triangular matrix:** A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if $a_{ij} = 0$ for all $i < j$

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

Operations of matrices

1. **Equality of matrices**:- Two matrices of same order are equal if their corresponding elements are equal.

Example: $\begin{bmatrix} 2x+4 & 3y-x \\ x+2 & x+z \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 3 & -1 \end{bmatrix}$

Find the value of $x+y+z$.

A) 0 B) 1 C) -2 D) -6

2. **Addition of matrices**:
Addition of two matrices is possible only if they are of same order.

The resultant matrix is obtained by adding of corresponding elements of two given matrices

$$\boxed{\begin{array}{l} A=[a_{ij}] \quad . \quad B=[b_{ij}] \\ \text{if } C=A+B \\ \Rightarrow [c_{ij}]=[a_{ij}]+[b_{ij}] \end{array}}$$

Example !
if $A=\begin{bmatrix} 2 & -1 \\ 0 & 6 \end{bmatrix}_{2\times 2}$ $B=\begin{bmatrix} 4 & 7 \\ 2 & -2 \end{bmatrix}_{2\times 2}$ Then: $A+B$

~~A+B~~ **Properties of matrix addition**

1. Matrix addition is commutative $A+B=B+A$
2. Matrix addition is associative $A+(B+C)=(A+B)+C$
3. Cancellation law holds i.e, if: $A+B=A+C$

3. Substraction of matrices.

Substraction of two matrices is possible only if they are of same order.

The resultant matrix is obtained by substraction of corresponding elements of two given matrices.

$$\boxed{\begin{array}{l} A = [a_{ij}] \quad , B = [b_{ij}] \\ \text{if} \quad C = A - B \\ \Rightarrow [c_{ij}] = [a_{ij}] - [b_{ij}] \end{array}}$$

4. Multiplication of matrix with a scalar.

The scalar gets multiplied in every element of the matrix.

Example: $A = \begin{bmatrix} 2 & -5 & 1 \\ 0 & 6 & -1 \end{bmatrix}_{2\times 3}$

then,

$$2A = \begin{bmatrix} 4 & -10 & 2 \\ 0 & 12 & -2 \end{bmatrix}$$

5. Multiplication of two matrices

Matrix multiplication is done Row by column.

for the two matrices $A_{m\times n}$ and $B_{p\times q}$. the multiplication $(A\times B)$ is possible only if number of columns of A is equal to the number of rows of B. i.e. $n = p$

The resultant matrix C will be of the order $m\times q$

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times 2}$ and $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 2 \end{bmatrix}_{2\times 3}$

Find AB and BA.

Properties of matrix multiplication.

1. In general, $AB \neq BA$ i.e commutative property does not hold. They may be equal for special set of matrices A and B.

2. Multiplication of matrices is distributive w.r.t a addition of matrices $A(B+C) = AB + AC$.

3. Matrix multiplication is associative if conformability is assured $A(BC) = (AB)C$.

4. If $AB = 0$ then it does not neccessarily mean that $A=0$ or $B=0$ or both are 0.

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5. $AI_n = I_nA = A$

Example:

if $A = \begin{bmatrix} a & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

then the value of a for which $A^2 = B$ is

A. 1 B. -1 C. 4 D. No real values

Example: Find the value of x if $[1 \; x \; 1] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = 0.$

The transpose of a matrix:

For the matrix A, the matrix obtained by interchanging rows and columns is called Transpose of matrix A and is denoted by A' or A^T

properties of transpose:

1. $(A')' = A$
2. $K(A') = (KA')$; K being a scalar.
3. $(A+B)' = A' + B'$
4. $(AB)' = B'A'$
5. $(ABC)' = C'B'A'$

Ex: if matrix A, $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ where a, b, c are real positive numbers, $abc = 1$ and $A^T A = I$, then find the value of $a^3 + b^3 + c^3$.

A) 6 B) 2 C) 4 D) 0

Ex: if P is a 3x3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3x3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that

A) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ B) $PX = X$ C) $PX = 2X$ D) P

* Trace of matrix *

if $A = [a_{ij}]_{m \times m}$ be a square matrix then the sum of its diagonal elements is defined as the trace of the matrix A. and is written as Tr(A)

properties of trace.

1. $Tr(pA) = p\,Tr(A)$
2. $Tr(pA + qB) = p\,Tr(A) + q\,Tr(B)$.
3. $Tr(AB) = Tr(BA)$, provided AB and BA

Some Special matrices.

1. Symmetric matrix.

Matrix in which $[a_{ij}] = [a_{ji}]$ is called symmetric matrix.

Here, $A' = A$.

2. Skew symmetric matrix :- A square matrix in which $[a_{ij}] = -[a_{ji}]$ is called skew-symmetric matrix.

Here, $A' = -A$.

properties of symmetric and skew-symmetric matrix

1. In skew symmetric matrix all the diagonal elements are zero.

2. for any matrix $(A+A')$ is always symmetric and $(A-A')$ is skew symmetric.

$B = A + A^T$	$C = A - A^T$
$B^T = (A+A^T)^T$	$C^T = (A-A^T)^T$
$B^T = A^T + (A^T)^T$	$C^T = A^T - A$
$B^T = A^T + A$.	$C^T = -(A - A^T)$

3. Any matrix can be written as sum of symmetric and skew symmetric matrix.

$$A = \frac{(A+A^T)}{2} + \frac{(A-A^T)}{2}$$

Ex:- let X and Y be two arbitrary, 3×3, non-zero, skew symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew-symmetric:

A. $Y^3Z^4 - Z^4Y^3$.

B. $X^{44} + Y^{44}$

C. $X^4Z^3 - Z^3X^4$

D. $X^{23} + Y^{23}$

Orthogonal matrix :- A square matrix 'A' is said to be orthogonal if $A^TA = AA^T = I$

$$\boxed{A^TA = AA^T = I}$$

Idempotent matrix : $A^2 = A$

$$\boxed{A^2 = A}$$

Involutory matrix : A matrix 'A' will be called an involutory matrix if $A^2 = I$ (unit matrix)

$$\boxed{A^2 = I}$$

periodic matrix : A matrix 'A' will be called an periodic matrix if $A^{K+1} = A$ where K is least positive integer for which $A^{K+1} = A$, then K is said to be the period of A.

$$\boxed{A^{K+1} = A \text{ where } K \text{ is period of } A}$$

Nilpotent matrix : A matrix 'A' will be called nilpotent matrix if $A^K = 0$ (null matrix), K is least positive integer and K is called index of the nilpotent matrix

$$\boxed{A^K = 0,\ K \text{ is index of } A}$$

Ex :- if A, B are two Idempotent matrices and $AB = BA = 0$ then $A+B$ is

A. Involutory matrix B. Idempotent matrix

C. orthogonal matrix D. Nilpotent matrix

Ans :- $A^2 = A$; $B^2 = B$

$$(A+B)^2 = (A+B)(A+B)$$
$$= A^2 + AB + BA + B^2$$
$$= A^2 + B^2$$

Ex:- Let A is a matrix of order 2x2 such that $A^2 = 0$. Then Tr(A) is:

A) 1   B) 0   C) −1.   D) None

Ans:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \quad A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$= b(a+d) = 0 \qquad a+d \neq 0 \Rightarrow b = 0 \;\&\; c = 0.$

$c(a+d) = 0 \qquad \because a^2 + bc = 0$

$\Rightarrow a = 0, \; bc + d^2 = 0 \Rightarrow d = 0$

— : DETERMINANTS : —

Determinant:

It is a number.

It is defined only for a square matrix.

Determinant of matrix A is written as $|A|$

Minors:-

minors of an element $[a_{ij}]$ is expressed as M_{ij} and is obtained by removing i^{th} row and j^{th} column

Ex:- Find minor M_{23} in the following terms

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & -1 & 2 \end{vmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$$

$-1 - 8 = -9.$

—: Cofactor :—

Cofactor of an element $[a_{ij}]$ is expressed as c_{ij} and is calculated as:

$$c_{ij} = (-1)^{i+j} M_{ij}$$

Ex:- Find cofactor C_{23}: $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & -1 & 2 \end{vmatrix}$

$$M_{23} = \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix}$$

$$M_{23} = -1 - 8 = -9.$$

$$C_{23} = (-1)^{2+3} M_{23}$$

$$C_{23} = -1 \times -9 = 9$$

Calculation of Determinant

It is the sum of product of elements of a row (or column) with their corresponding cofactors.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Ex:- Evaluate the following determinants:

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & -1 & 2 \end{vmatrix}$$

Ex:- if $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x+iy$; then

A. $x=3, y=1$ B. $x=1, y=3$

C. $x=0, y=3$ D. $x=0, y=0$

Multiplication of scalar to a determinant

The scalar (number) gets multiplied in any one row or column.

Ex:- let $P=[a_{ij}]$ be a 3×3 matrix and let $Q=[b_{ij}]$ where $[b_{ij}] = 2^{i+j}[a_{ij}]$ for $1 < i, j < 3$. if the determinants of P is 2. then the determinants of the matrix of Q is.

A 2^{10} B. 2^{11} C. 2^{12} D. 2^{13}.

$$|P| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow |Q| = 2^{i+j}|P|$$

$$= \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix}$$

$$|Q| = (2^2)(2^3)(2^4)(2)(2^2)\,|P|$$

$$|Q| = 2^{12} \times |P|$$

$$= 2^{12} \times 2$$

$$|Q| = 2^{13}$$

properties of Determinants :—

1. Determinant is zero if any row or column is zero.
2. Determinant is zero if two rows or column have equal or proportional values.
3. Value of determinant changes by a minus sign if we exchange two rows or two columns.
4. Elementary row or column transformation do not change the value of determinant

$R_1 \to R_1 + R_3 \to R_3 + mR_2$.

5. $|AB| = |A|\,|B|$
6. $|A^T| = |A|$
7. $|KA| = K^n|A|$; n is order of A.

Important results.

1. Determinant of skew symmetric matrix of odd order is zero.
2. Determinant of diagonal matrix is equal to the product of its diagonal elements.
3. If a number is multiplied to a determinant, it gets multiplied in any one row or column.

Singular and non-singular matrix:

Matrix whose determinant is zero is called singular matrix otherwise it is non-singular matrix.

Ex:- $\begin{vmatrix} 2c & 2c & c-a-b \\ a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \end{vmatrix}$

A. $(a+b+c)^2$ B) $(a+b+c)^3$ C) $(a+b+c)$ D) $(a+b+c)^4$

Ex:- The determinant $\begin{vmatrix} a & b & aa+b \\ b & c & ba+c \\ aa+b & ba+c & 0 \end{vmatrix}$ is equal to zero if

A. a, b, c are in AP

B. a, b, c are in GP

C. a, b, c are in HP

D. a is a root of the equation $ax^2+2bx+c$.

Ex: if a, b, c are sides of a triangle and

$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 0$, then

A. ΔABC is equilateral

B. ΔABC is a Isosceles

C. ΔABC is a right angled triangle

D. ΔABC is a Scalene triangle.

—: Inverse of a Matrix :—

let A be any n-rowed square matrix. Then a matrix B, if exists such that $AB = BA = I_n$ is called the inverse of A.

inverse of A is usually denoted by A^{-1} (if exists)

$$A(\text{adj}\,A) = |A|\, I$$

$$A\left(\frac{1}{|A|}\,\text{adj}\,A\right) = I$$

properties of Inverse matrix

1. Every invertible matrix posses a unique inverse
2. if A and B are invertible matrices of the same order, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
3. if A is invertible square matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$
4. $|A^{-1}| = |A|^{-1}$

Ex: if $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1}

Ex: $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & a \\ 1 & -2 & 3 \end{bmatrix}$

if B is inverse of A then a is :-

Inverse using elementary transformation

Ex:- if $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1}

Cayley - Hamilton Theorem.

Every square matrix satisfies a specific polynomial equation known as characteristic equation.

$$P(\lambda) = |A - \lambda I|$$

$$P(A) = 0$$

$$a_1 \lambda^3 + b_1 \lambda^2 + c_1 \lambda + d_1 = 0$$

$$a_1 A^3 + b_1 A^2 + c_1 A + d_1 I = 0$$

Ex! if $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, prove that $A^2 - 4A - 5I = 0$

Ex:- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$$A^{-1} = \left[\frac{1}{6}(A^2 + cA + dI)\right]$$

Then the value of c and d are:

A $(-6, -11)$. B. $(6, 11)$. C $(-6, 11)$ D $(6, -1$

S. System of linear simultaneous Equations:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

I. $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ Unique Solution

II. $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ No solution

III. $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ Infinite solution.

Ex: The number of values of K for which the system of equations $(K+1)x + 8y = 4K$; $Kx + (K+3)y = 3K-1$ has Infinitely many solution is.

A 0 B 1 C 2 D Infinite

Ans: $(K+1)x + 8y = 4K$
$Kx + (K+3)y = 3K-1$

$$\frac{K+1}{K} = \frac{8}{K+3} = \frac{4K}{3K-1}$$

to get ans: $K=1$ & $K=3$.

System of linear simultaneous Equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$\Rightarrow AX = B$

Method 1 Matrix method

a) $\Delta \neq 0$, then A^{-1} exists

$\Rightarrow A^{-1}(AX) = A^{-1}B$

$\Rightarrow X = A^{-1}B$

and therefore unique values of x, y and z are obtained.

b) $\Delta = 0$ then A^{-1} does not exists

We have $AX = B$

$\Rightarrow ((\text{adj } A)A)X = (\text{adj } A)B$

$\Rightarrow \Delta X = (\text{adj } A)B$.

1. $(\text{adj } A)B = 0$, then the system $AX = B$ has infinitely many solution.

2. $(\text{adj } A)B \neq 0$ then the system $AX = B$ has no solution.

Determinant method of solution.

$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
$a_3x + b_3y + c_3z = d_3$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\boxed{AX = B}$$

method 2 (Cramer's Rule)

let $|A| = \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

obtained on replacing first column of Δ by B

Similiarly, $\Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $\Delta z = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$x \cdot \Delta = \Delta x \qquad y \cdot \Delta = \Delta y \qquad z \cdot \Delta = \Delta z$

Case 1

if $\Delta \neq 0$, then the given systems of equations has unique solutions given by

$$x = \frac{\Delta x}{\Delta}, \quad y = \frac{\Delta y}{\Delta} \text{ and } z = \frac{\Delta z}{\Delta}$$

Case 2

if $\Delta = 0$, then two sub cases arise :

a) At least one of Δx, Δy and Δz is non-zero

$$x \cdot \Delta = \Delta x \quad y \cdot \Delta = \Delta y, \quad z \cdot \Delta = \Delta z$$

(No solution)

Case 3

if $\Delta = 0$, then two sub cases arise

b) All of Δx, Δy and Δz are zero

Either No solution or Infinite solution

Ex:- Given

$$2x - y + 2z = 2$$
$$x - 2y - z = -4$$
$$x + y + \lambda z = 4$$

then the value of λ, such that the given system of equation has No solution

A. 3 B. 1 C. 0 D. -1

Solution of Homogeneous simultaneous equation

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Ex: The system of equation.

$$\lambda x + y + z = 0$$
$$-x + \lambda y + z = 0$$
$$-x - y + \lambda z = 0$$

will have a non-zero solution if real values of λ are given by:

$$\Delta = \begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$$

$$\lambda(\lambda^2+1) - 1(-\lambda+1) + 1(1+\lambda) =$$
$$\lambda^3 + \lambda + \lambda - 1 + 1 + \lambda = 0$$
$$\lambda^3 + 3\lambda = 0$$