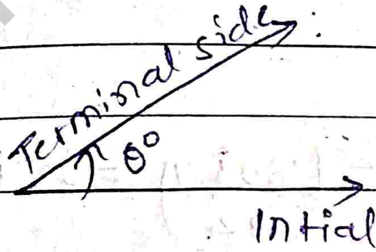
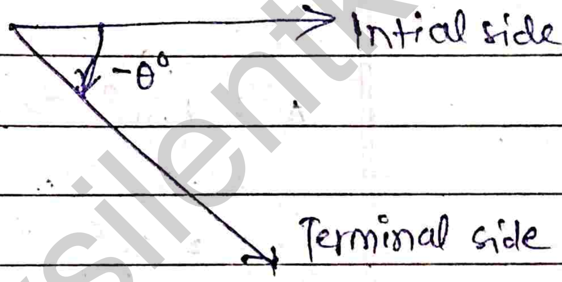
Welcome to 2nd semester.

UNIT - 1 Trigonometry. MATHEMATICS.

Trigonometry - The branch of mathematics which deals with the measurements of angles of triangles and the problems related to these angles.

Measure of angle → The amount of rotation from the Intial side to the terminal side is called Measure of the angle.





• Sexagesimal system (Degree Measure) - The angle traced by a moving line about a point from its intial position to the terminating position in making $\frac{1}{360}$ of the complete revolution of a circle.
It is said to be 1° (1 degree).

$$1' = \frac{1^\circ}{60'}$$

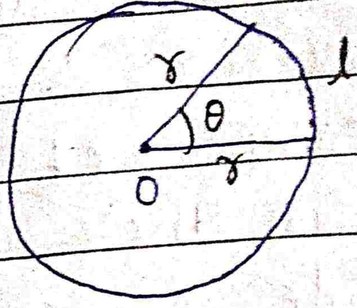
$$1'' = \frac{1'}{60'}$$

$$\boxed{1 \text{ right angle} = 90^\circ}$$

$$\boxed{1^\circ = 60'} \text{ and } \boxed{1' = 60''}.$$

(ii) Circular system (Radian measure) —
A radian is an angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.

$$\boxed{\theta = \frac{l}{r}}$$



$$\boxed{\pi^c = 180^\circ}$$

$$\pi^c = 180^\circ \Rightarrow 1^c = \left(\frac{180}{\pi}\right)^\circ = \left(\frac{180 \times 7}{22}\right)^\circ = 57^\circ 16' 21''$$

$$180^\circ = \pi^c = 1^\circ = \left(\frac{\pi}{180}\right)^c = \left(\frac{22}{7} \times \frac{1}{180}\right)^c = 0.01746^c$$

(i) In the Degree measure, we measure angles in degree, minutes & seconds.

1 right angle $= 90^\circ$
$1^\circ = 60'$
$1' = 60''$

(ii) In the circular measure, we measure angles in radians. $\pi^c = 180^\circ$

(iii) if an arc of length l makes an angle θ^c at the centre of a circle of radius r.
$\theta = \frac{l}{r}$.

Q. $\left(\frac{7\pi}{12}\right)^c =$ $\quad 1^c = \left(\frac{180}{\pi}\right)^\circ$

$$\left(\frac{7\pi}{12}\right)^c = \left(\frac{\cancel{180}^{15}}{\cancel{\pi}} \times \frac{7\cancel{\pi}}{\cancel{12}}\right)^\circ = 105^\circ.$$

Q. $15^\circ =$ $\quad 1^\circ = \left(\frac{\pi}{180}\right)^c$

$$= \left(\frac{\pi}{\cancel{180}_{12}} \times \cancel{15}\right)^c = \left(\frac{22}{7} \times \frac{1}{180} \times 15\right)^c = \left(\frac{\pi}{12}\right)^c.$$

$=$

$\left(\frac{3}{4}\right)^c$ $\quad 1^c = \left(\frac{180}{\pi}\right)^\circ$

$$\frac{3}{4}^c = \left(\frac{180}{\pi} \times \frac{3}{4}\right)^\circ = \left(180 \times \frac{7}{22} \times \frac{3}{4}\right)^\circ$$

$$= 42^\circ 57' 16''.$$

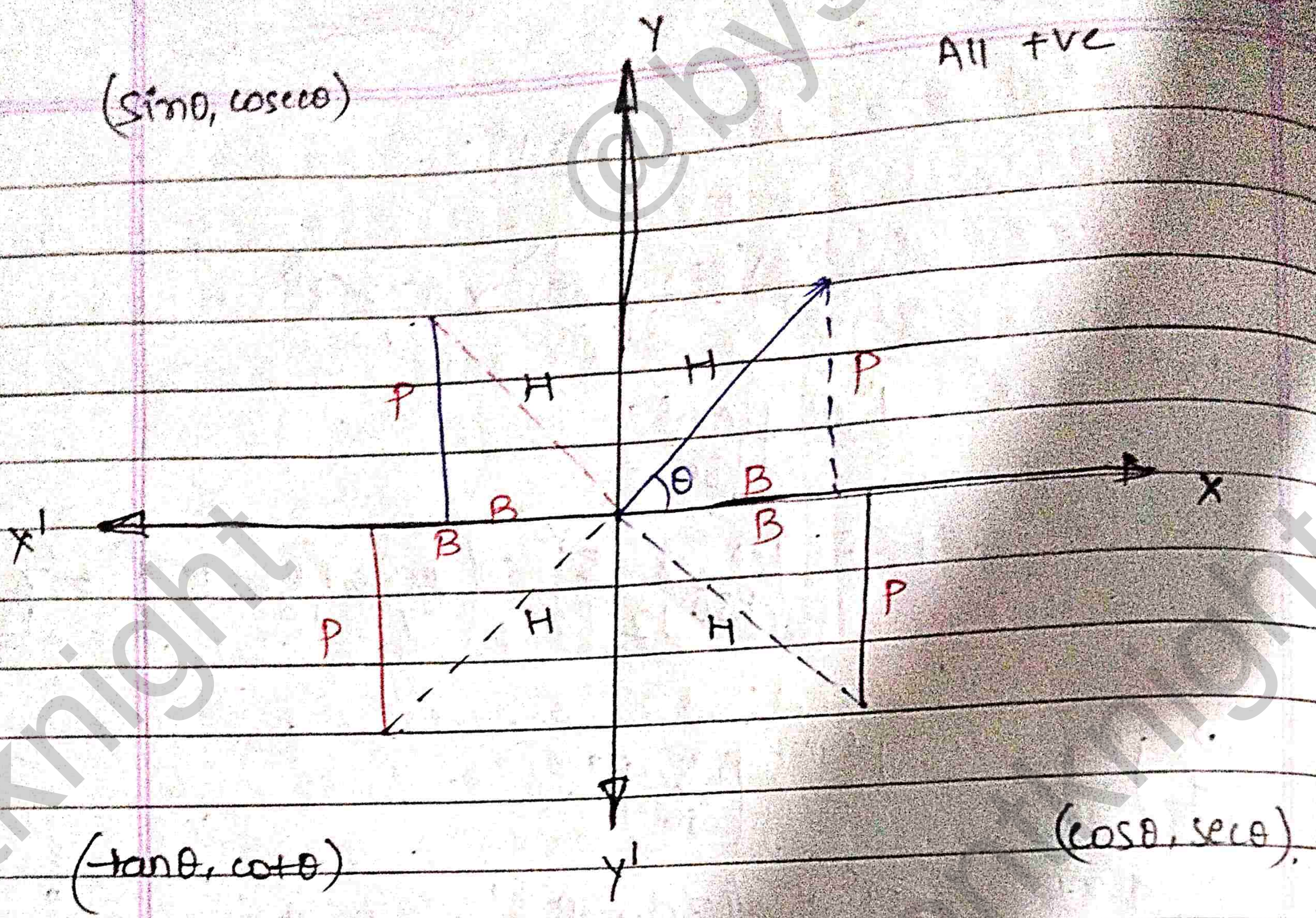
$-37^\circ 30' \Rightarrow 1^\circ = \left(\frac{\pi}{180}\right)^c$

$$-37^\circ 30' = \left\{\frac{\pi}{180} \times -37^\circ 30'\right\}^c$$

$$= \frac{\pi}{180} \times \left(\frac{37\frac{1}{2}}{2}\right)^\circ$$

$$= -\frac{\pi}{180} \times \frac{75}{2}^\circ = \left(-\frac{5\pi}{24}\right)^c$$

Hence, $(-37^\circ 30') = \left(\frac{-5\pi}{24}\right)^c.$



$$\sin\theta = \frac{P}{H}, \quad \operatorname{cosec}\theta = \frac{H}{P}$$

$$\cos\theta = \frac{B}{H}, \quad \sec\theta = \frac{H}{B}$$

$$\tan\theta = \frac{P}{B}, \quad \cot\theta = \frac{B}{P}$$

$$\sec\theta = \frac{1}{\cos\theta}, \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \tan\theta = \frac{1}{\cot\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \cot^2\theta = \operatorname{cosec}^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$

Negative Arc length.

$\sin(-\theta) = -\sin\theta$, $\tan(-\theta) = -\tan\theta$

$\boxed{\cos(-\theta) = \cos\theta}$, $\boxed{\sec(-\theta) = \sec\theta}$

$\sec(-\theta) = \sec\theta$, $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$.

- $\sin(2n\pi+\theta) = \sin\theta$, $\operatorname{cosec}(2\pi n+\theta) = \operatorname{cosec}\theta$
- $\cos(2n\pi+\theta) = \cos\theta$, $\sec(2n\pi+\theta) = \sec\theta$
- $\tan(n\pi+\theta) = \tan\theta$, $\cot(n\pi+\theta) = \cot\theta$

①

- $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$, $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$.
- $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$, $\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$.
- $\sec\left(\frac{\pi}{2}-\theta\right) = \operatorname{cosec}\theta$, $\operatorname{cosec}\left(\frac{\pi}{2}-\theta\right) = \sec\theta$

②

$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$, $\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$

$\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$, $\cot\left(\frac{\pi}{2}+\theta\right) = -\tan\theta$

$\sec\left(\frac{\pi}{2}+\theta\right) = -\operatorname{cosec}\theta$, $\operatorname{cosec}\left(\frac{\pi}{2}+\theta\right) = \sec\theta$

③

$\cos(\pi-\theta) = -\cos\theta$

$\sin(\pi-\theta) = \sin\theta$

$\tan(\pi-\theta) = -\tan\theta$

④

$\cos(\pi+\theta) = -\cos\theta$

$\sin(\pi+\theta) = -\sin\theta$

$\tan(\pi+\theta) = \tan\theta$

5.

$$\cos(2\pi-\theta)=\cos\theta$$
$$\sin(2\pi-\theta)=-\sin\theta$$
$$\tan(2\pi-\theta)=-\tan\theta$$

6.

$$\cos(2\pi+\theta)=\cos\theta$$
$$\sin(2\pi+\theta)=\sin\theta$$
$$\tan(2\pi+\theta)=\tan\theta$$

7.

$$\cos\left(\frac{3\pi}{2}-\theta\right)=-\sin\theta$$
$$\sin\left(\frac{3\pi}{2}-\theta\right)=-\cos\theta$$

8.

$$\cos\left(\frac{3\pi}{2}+\theta\right)=\sin\theta$$
$$\sin\left(\frac{3\pi}{2}+\theta\right)=-\cos\theta$$

यदि Trigonometry function मे angle हो, तो $(90-\theta)$, $(90+\theta)$, $(270-\theta)$, $(270+\theta)$ function change करेगा, अथवा $(180-\theta)$, $(180+\theta)$, $(360-\theta)$, $(360+\theta)$ हो, तो function change नहीं करेगा।

* Trigonometric functions of sum and difference of numbers.

1. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
   $\cos(A-B) = \cos A\cos B + \sin A\sin B$.

2. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
   $\sin(A-B) = \sin A\cos B - \cos A\sin B$

3. $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A\cdot\tan B}$, $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\cdot\tan B}$

4. $$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

$$\cot(A-B) = \frac{\cot A \cdot \cot B + 1}{\cot B - \cot A}$$

5. $$\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$$
$$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$$

Some more Important Functions—

- $2\sin A \cos B = \sin(A+B) + \sin(A-B)$.
- $2\cos A \sin B = \sin(A+B) - \sin(A-B)$.
- $2\cos A \cos B = \cos(A+B) + \cos(A-B)$.
- $2\sin A \sin B = \cos(A+B) - \cos(A+B)$.

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Trigonometric function of multiples of 2x

$$\sin 2x = 2\sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}$$

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\cos 3x = 4\cos^3 x - 3\cos x$$

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}$$

$$\sin 18^\circ = \frac{\sqrt{5}-1}{4} = \cos 72^\circ$$

$$\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72^\circ$$

$$\cos 36^\circ = \frac{\sqrt{5}+1}{4} = \sin 54^\circ$$

$$\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54^\circ$$

Trigonometric functions of half angles.

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, $1-\cos\theta = 2\sin^2\frac{\theta}{2}$

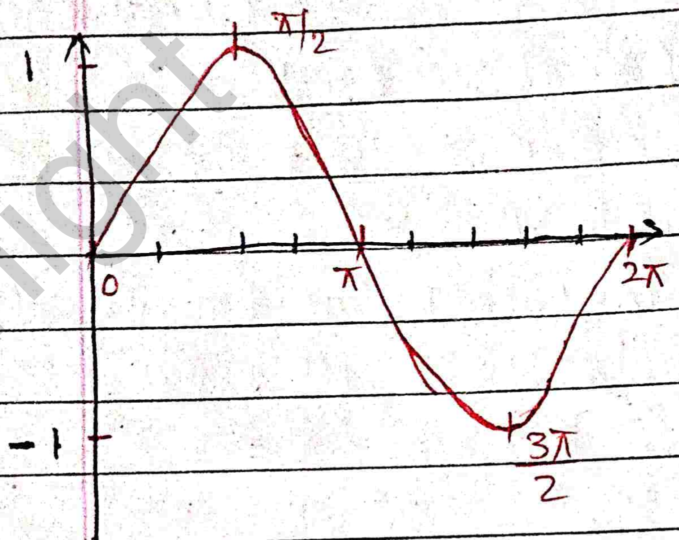
$1+\cos\theta = 2\cos^2\frac{\theta}{2}$, $\sin\theta = \frac{2\tan\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$

$\cos\theta = \frac{1-\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}$

Graph

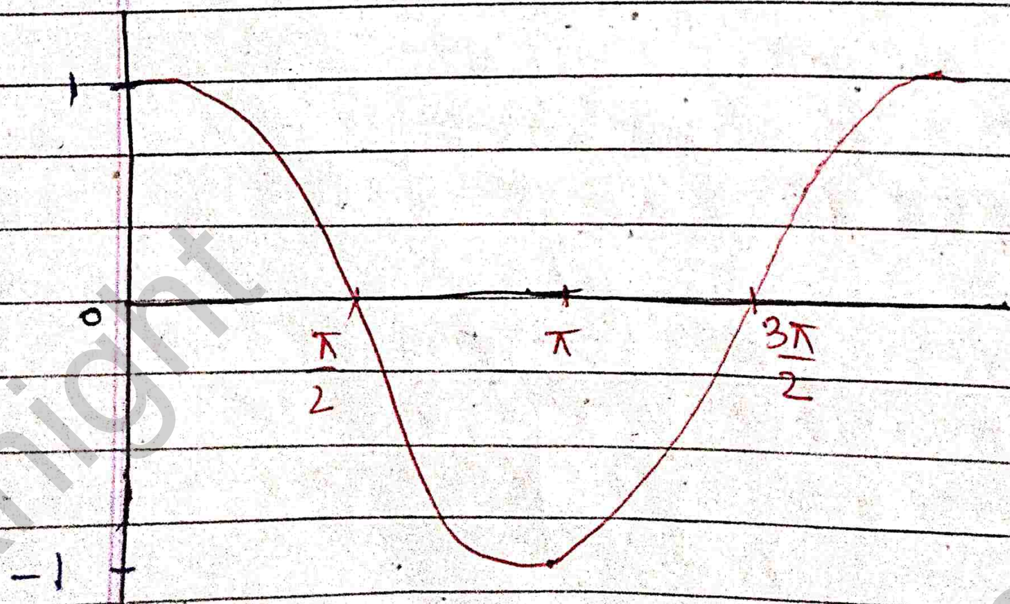
$y = \sin x$

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$3\pi/2$	2π
$\sin x$	0	$1/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/2$	0	$-1/2$	-1	0



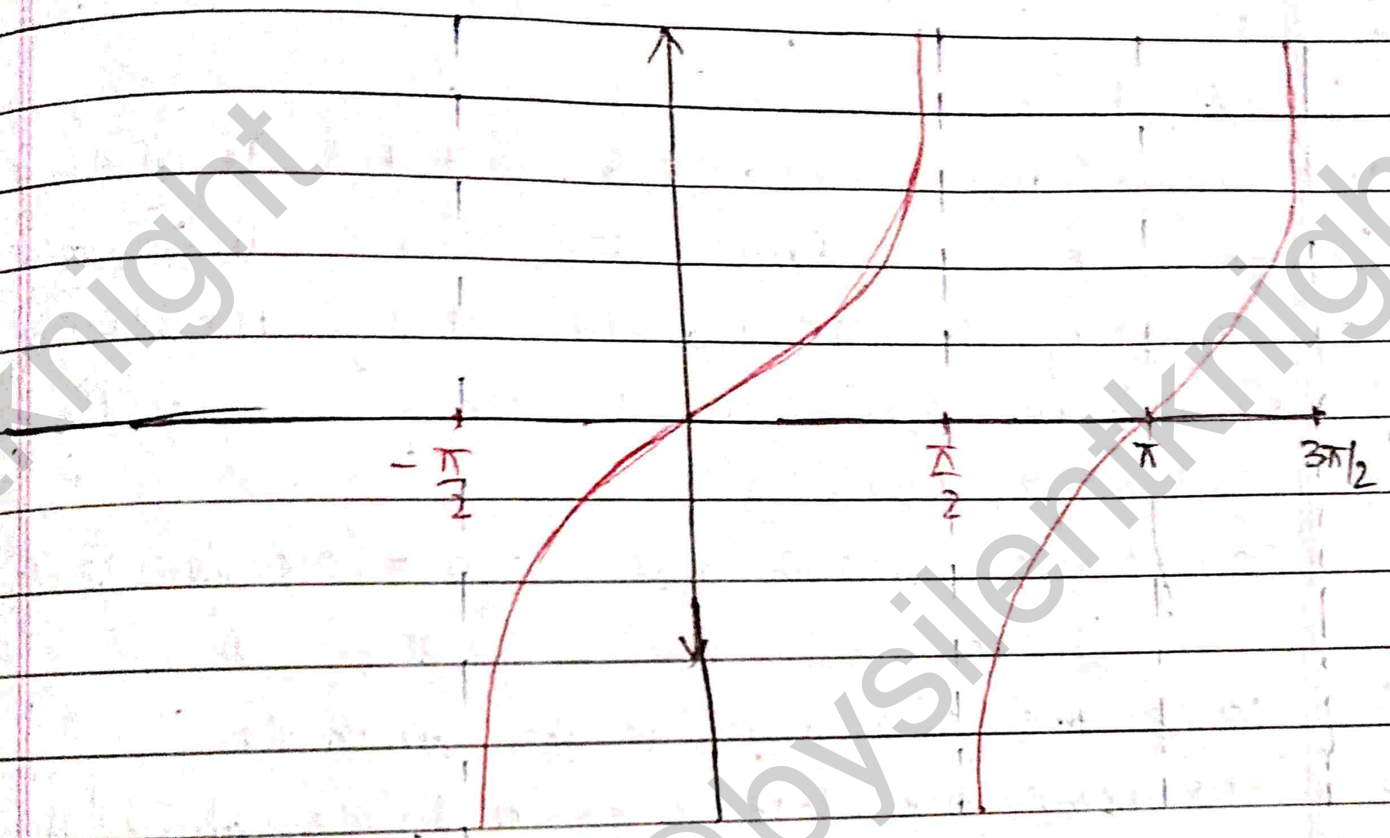
$y = \cos x$

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$	$3\pi/2$	2π
$\cos x$	1	$\sqrt{3}/2$	$1/2$	0	$-1/2$	$-\sqrt{3}/2$	-1	$-\sqrt{3}/2$	0	1



$y = \tan x$

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\tan x$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-1/\sqrt{3}$	0



Conditional Identities Involving the angles of a triangle

TYPE 1 Identities Involving sines and cosines.

Method –

1. Express the sum of the first two term as a product
2. In this product, Express the sum of two angles in terms of the third angle, using $A+B+C=\pi$
3. Expand the third term by using one of the following relations:

$\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = (2\cos^2\theta - 1), = (1-2\sin^2\theta)$

4. Take the common factor outside
5. Express the T-ratio of a single angle into a sum of two angles, and use the neccessary formula from the ones given in the box.

Q. if $A+B+C=\pi$, prove that $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$.

Sol^n:– L.H.S $(\sin 2A + \sin 2B) + \sin 2C$

$$2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) + \sin 2C$$

$$2\sin(A+B)\cos(A-B) + 2\sin C\cos C$$

$$2\sin(\pi - C)\cos(A-B) + 2\sin C\cos C$$

$$2\sin C\cos(A-B) + 2\sin C\cos C.$$

$$2\sin C\left[\cos(A-B) + \cos C\right]$$

$$2\sin C\left[\cos(A-B) + \cos[\pi-(A+B)]\right] \quad \left[\because A+B+C=\pi,\ C=\pi-(A+B)\right.$$

$2\sin C\left[\cos(A+B) - \cos(A+B)\right]$

$2\sin C\left[2\sin\left(\frac{A+B-A+B}{2}\right)\sin\left(\frac{A+B+A-B}{2}\right)\right]$

$2\sin C \cdot 2\sin B \cdot \sin A$

$= 4\sin A\sin B\sin C.$

Q. If $A+B+C=\pi$ prove that $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C.$

TYPE 2 Identities Involving squares of sines and cosines

Method -

change the squares of sines and cosines into cosines of double the angles by using the formula.

$\cos^2\theta = \frac{(1+\cos 2\theta)}{2}$; $\sin^2\theta = \frac{(1-\cos 2\theta)}{2}$.

Q. If $A+B+C=\pi$, prove that $\sin^2 A + \sin^2 B + \sin^2 C = 2(1+\cos A\cos B\cos C)$.

Sol^n - $\sin^2 A + \sin^2 B + \sin^2 C$

$\frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} + \frac{1-\cos 2C}{2}$

$= \frac{1-\cos 2A + 1 - \cos 2B + 1 - \cos 2C}{2}$

$= \frac{3-(\cos 2A + \cos 2B + \cos 2C)}{2}$

$$\frac{3}{2} - \frac{1}{2}(\cos 2A + \cos 2B + \cos 2C)$$

$$\frac{3}{2} - \frac{1}{2}\left[2\cos\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right) + \cos 2C\right]$$

$$\frac{3}{2} - \frac{1}{2}\left[2\cos(A+B)\cos(A-B) + \cos 2C\right]$$

$$\frac{3}{2} - \frac{1}{2}\left[2\cos(\pi - C)\cos(A-B) + 2\cos^2 C - 1\right]$$

$$\frac{3}{2} - \frac{1}{2}\left(-2\cos C\cos(A-B) + 2\cos^2 C - 1\right)$$

$$\frac{3}{2} - \frac{1}{2}\left[-2\cos C\{\cos(A-B) - \cos C\}\right] + \frac{1}{2}$$

$$\frac{3}{2} - \frac{1}{2}\left[-2\cos C\{\cos(A-B) + \cos(\pi-(A+B))\}\right] + \frac{1}{2}$$

$$\frac{3}{2} - \frac{1}{2}\left[-2\cos C\{\cos(A-B) + \cos(A+B)\}\right] + \frac{1}{2}$$

$$\frac{3}{2} - \frac{1}{2}\left[-\cancel{2}\cos C\{2\cos A\cos B\}\right] + \frac{1}{2}$$

$$2 - \frac{1}{\cancel{2}} \times -\cancel{4}\cos A\cos B\cos C$$

$$2 + 2\cos A\cos B\cos C$$

$$2(1 + \cos A\cos B\cos C)$$ proved.

3. Identities Involving tangents —

Method —

1. Using $A+B+C=\pi$, Express the sum of two angles in term of the third.

2. Take tangents on both sides and Expand LHS.
3. cross multiply and transpose.

Q. if $A+B+C=\pi$, P.T $\tan A+\tan B+\tan C=\tan A \cdot \tan B \cdot \tan C$.

Sol^n —
$$A+B=\pi-C$$
$$\tan(A+B)=\tan(\pi-C)$$
$$\frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$$
$$\tan A+\tan B=-\tan C+\tan A\cdot\tan B\cdot\tan C$$
$$\tan A+\tan B+\tan C=\tan A\cdot\tan B\tan C.$$

*J.M.SA

Civil Engineering

UNIT-2 Co-ordinate Geometry

MATHEMATICS

STRAIGHT LINES

1. Distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by—

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2. Area of ΔABC whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by—

$$\Delta = \frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right| \text{ sq units}$$

Note → The points A, B, C are collinear then, Area of $\Delta ABC = 0$.

3. (i) If the point $P(x, y)$ divides the join of $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m:n$.

$$x = \frac{mx_2 + nx_1}{m+n} \quad \text{and} \quad y = \frac{my_2 + ny_1}{m+n}$$

(ii) If $P(x, y)$ is the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$.

$$x = \frac{1}{2}(x_1 + x_2) \quad \text{and} \quad y = \frac{1}{2}(y_1 + y_2).$$

1. Find the distance between the points (2, -3) and (-6, 3)

A (2, -3) and B (-6, 3)

$AB = \sqrt{(-6-2)^2 + (3-(-3))^2}$

$= \sqrt{(-8)^2 + (6)^2} = 10$ unit.

. find the area of the triangle whose vertices are A (4, 4), B (3, -16) and C (3, -2).

$A\begin{pmatrix} x_1 = 4 \\ y_1 = 4 \end{pmatrix}$ $B\begin{pmatrix} x_2 = 3 \\ y_2 = -16 \end{pmatrix}$ $C\begin{pmatrix} x_3 = 3 \\ y_3 = -2 \end{pmatrix}$

$\Delta = \frac{1}{2} | x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) |$

$= \frac{1}{2} | 4(-16+2) + 3(-2-4) + 3[4-(-16)]$

$= \frac{1}{2} | 4 \times -14 + 3 \times -6 + 3 \times 20 |$

$= \frac{1}{2} (-56 - 18 + 60) = \frac{1}{2} | -74 + 60 | = | \frac{-14}{2} | = 7$ sq. u.

Q. find the co-ordinates of the point which divides the line segment joining the points A(5, -2) and B(9, 6) in the ratio 3:1.

$x = \cancel{3 \times 9} \cdot \frac{m x_2 + n x_1}{m+n}$; $y = \frac{m y_2 + n y_1}{m+n}$

$= \frac{3 \times 9 + 1 \times \cancel{+8}}{3+1}$; $y = \frac{3 \times 6 + 1 \times -2}{3+1}$

$y = \frac{16}{4} = 4$.

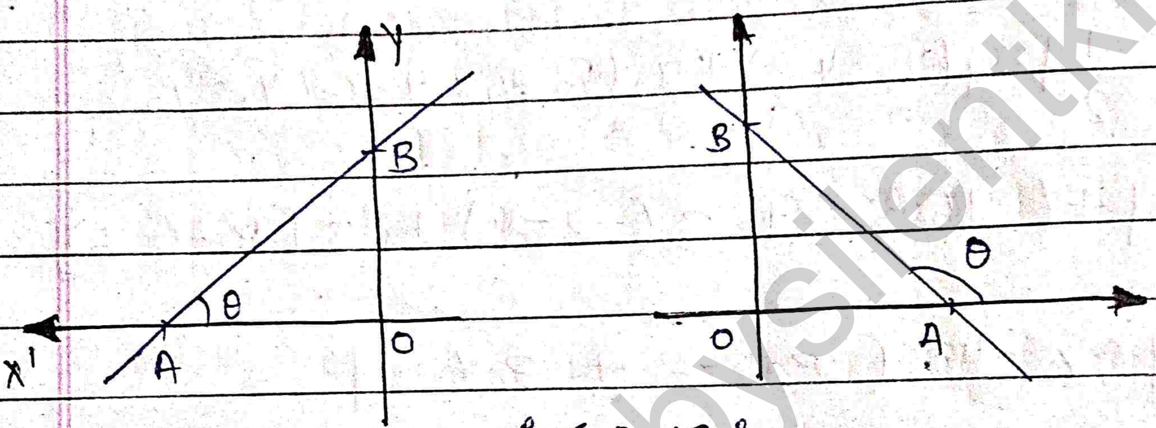
$= \frac{32}{4} = 8$

Q. find the co-ordinates of the midpoint of the line segment joining the points A(-2,-5) and B(3,-1).

Sol^n:- $x = \frac{(-2+3)}{2} = \frac{1}{2}$; $y = \frac{-5+(-1)}{2} = -3$

the required point is $\left(+\frac{1}{2}, -3\right)$.

— Slope of a line —



$0° \le \theta < 180°$.

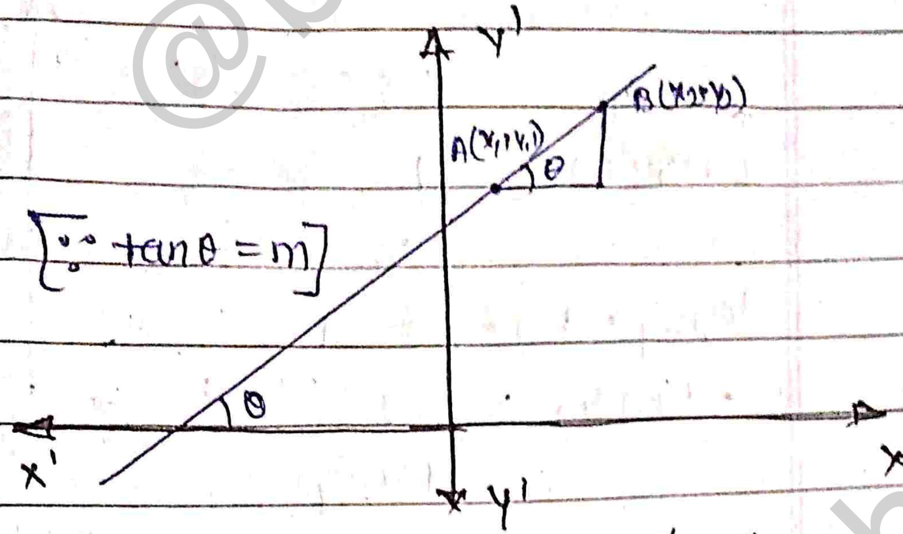
- Horizontal line - Any line parallel to the X-axis or the axis itself is called a horizontal line.
- Vertical line - Any line parallel to y-axis or the y-axis itself is called a vertical line.
- oblique line - A line which is neither horizontal nor vertical is called oblique line.

Slope or gradient of a line -

$$\boxed{m = \tan\theta}$$

- slope of a line passing through two given points.

$$\tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$



$[\because \tan\theta = m]$

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

Q. if the slope of line passing through the points $(2,5)$ and $(x,3)$ is 2.

slope of AB $= \dfrac{3-5}{x-2}$

$$\cancel{2} = \frac{-\cancel{2}}{x-2} \Rightarrow x-2 = -1$$

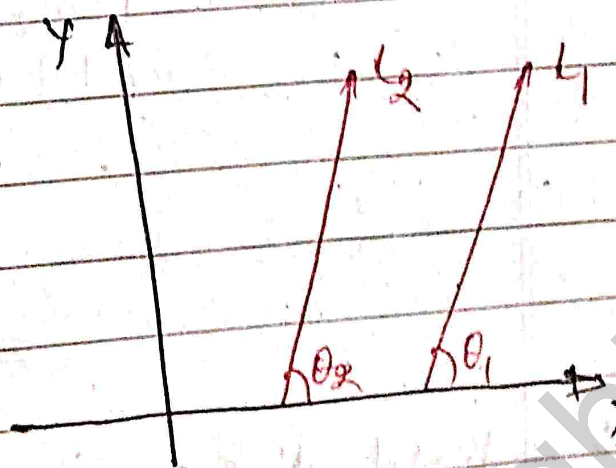
$$x = -1+2$$

$$\boxed{x = 1}.$$

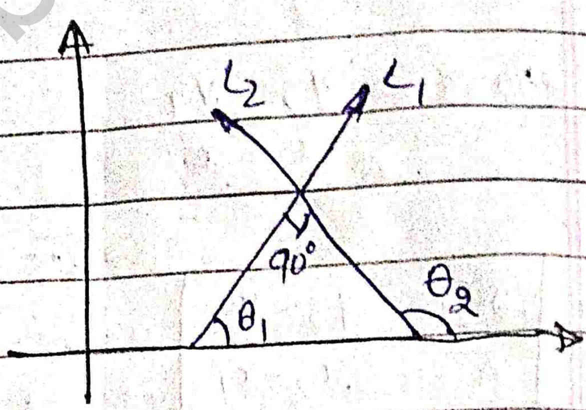
- slopes of parallel lines.

$$m_1 = m_2$$

l_1 and l_2 are parallel.



- Slopes of perpendicular lines—



$$\boxed{m_1 \cdot m_2 = -1}$$

$$\tan\theta_1 \cdot \tan\theta_2 = -1$$

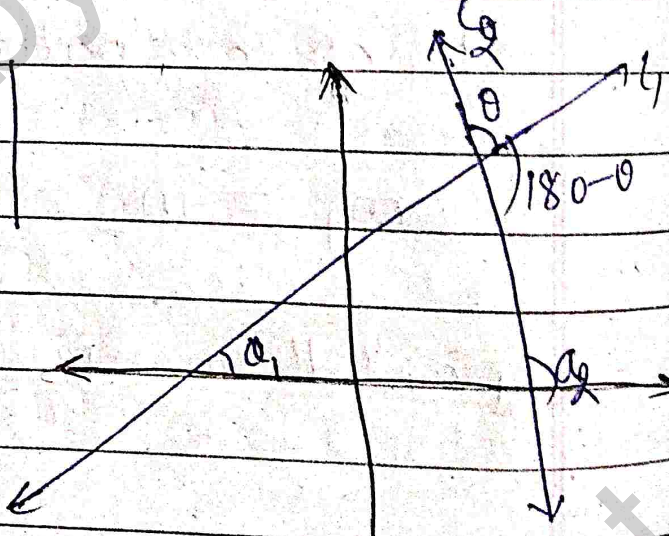
$$\tan\theta_2 = \frac{-1}{\tan\theta_1}$$

L_1 and L_2 are perpendicular to each other.

(i)	$L_1 \parallel L_2$	$\Leftrightarrow$	$m_1 = m_2$
(ii)	$L_1 \perp L_2$	$\Leftrightarrow$	$m_1 \cdot m_2 = -1$

- Angle between two Non-vertical and Non-perpendicular lines.

$$\tan\alpha = \left|\frac{m_2 - m_1}{1 + m_1 \cdot m_2}\right|$$



Q. find the angle b/w the lines whose slopes are $1/2$ and 3.

$m_1 = \frac{1}{2}, m_2 = 3$

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \Rightarrow \tan\theta = \left|\frac{3 - 1/2}{1 + 3 \cdot \frac{1}{2}}\right|$$

$$\tan\theta = \left|\frac{5/2}{5/2}\right| \Rightarrow \tan\theta = 1 \text{ or } \boxed{\theta = \pi/4}.$$

Equation of a line –

The linear relation between two variables x and y , which is satisfied by the co-ordinates of each and every points on the line and not by those of any other point in the cartesian plane.

(i) EQUATION OF X-AXIS.

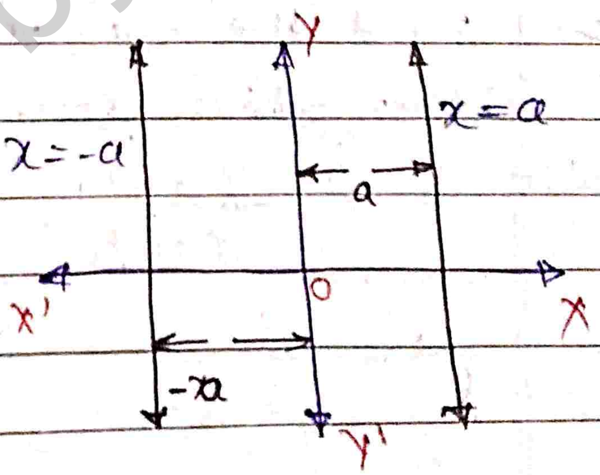
if $P(x, y)$ is any point on the x-axis then $y = 0$.

(ii) EQUATON OF Y-AXIS.

if $P(x, y)$ is any point on the y-axis then $x = 0$

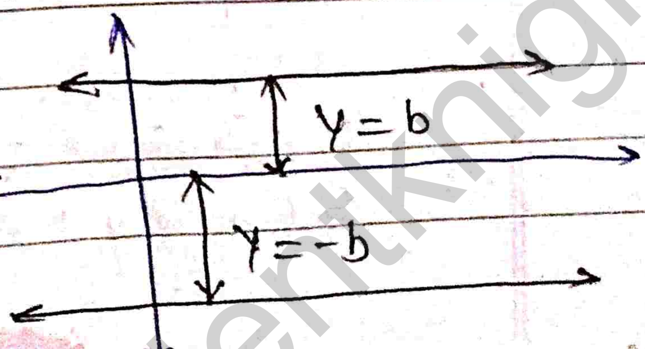
(iii) EQUATION OF A LINE PARALLEL TO Y-AXIS.

$x = a$, $x = -a$



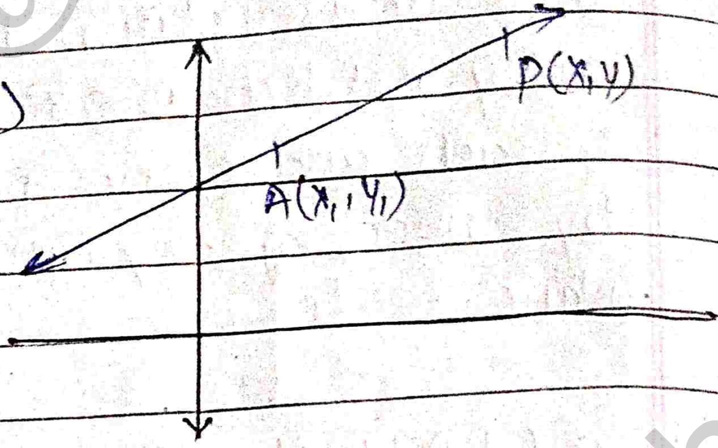
(iv) EQUATION OF A LINE PARALLEL TO X-AXIS.

$y = b$, $y = -b$



Equation of a line in point-slope form

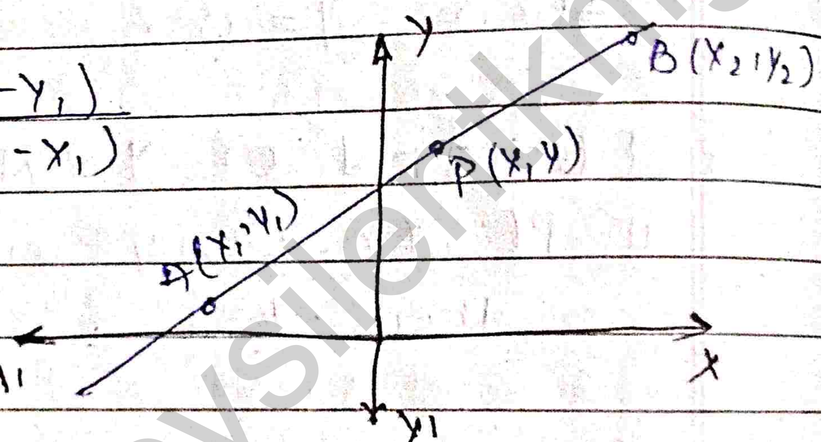
$$(y - y_1) = m(x - x_1)$$



$$\boxed{m = \frac{y - y_1}{x - x_1}}$$

Equation of a line in two point form

$$\frac{(y - y_1)}{(x - x_1)} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$



Q. Find the equation of a line passing through the points $(-1, 1)$ and $(2, -4)$.

Soln:-

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

or, $$\frac{y - 1}{x + 1} = \frac{-4 + 1}{2 + 1}$$

$$\frac{y - 1}{x + 1} = \frac{-5}{3}$$

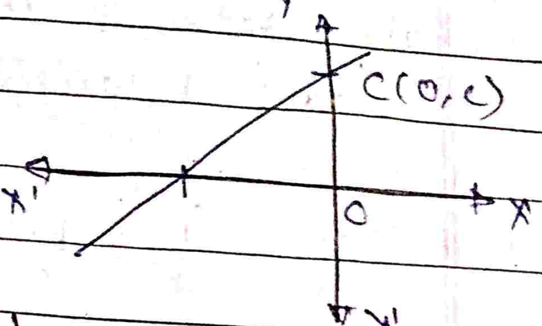
$$3y - 3 = -5x + 5$$

$$5x + 3y + 2 = 0.$$

Equation of a line in slope-Intercept form.

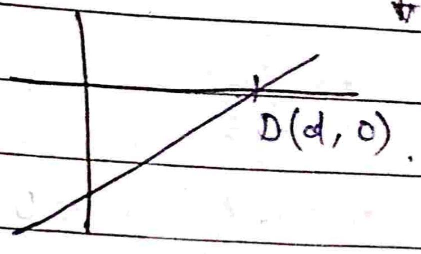
→ y-intercept c is —

$$y = mx + c.$$



→ x-intercept d is —

$$y = m(x - d).$$



Q: find the equation of a line whose slope is $\frac{1}{2}$ and y-intercept equal to $-5/4$.

$$y = mx + c \qquad m = \frac{1}{2},\ c = \frac{-5}{4}$$

$$y = \frac{1}{2}x + \left(\frac{-5}{4}\right)$$

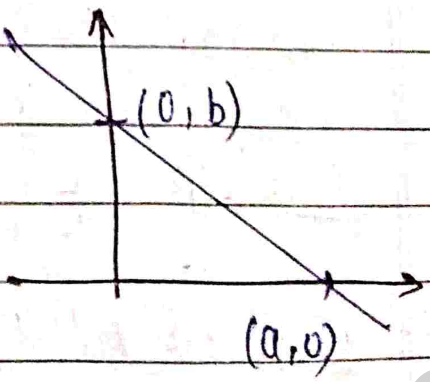
$$y = \frac{1}{2}x - \frac{5}{4} \rightarrow 4y = 2x - 5$$

$$2x - 4y - 5 = 0.$$

→ Equation of a line in intercepts form—

- The line making intercept a and b on the x- and y-axis.

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$



Q. find the equation of line which make intercept 2 & −3 on the x-axis and y-axis respectively.

$$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{x}{2} + \frac{y}{-3} = 1 \Rightarrow \frac{-3x + 2y}{-6} = 1$$

$$-3x + 2y = -6 \Rightarrow 3x - 2y - 6 = 0.$$

Q. find the equation of the line which makes intercepts 2 and -3 on the x-axis and y-axis respectively.

Sol^n – $a = 2,\ b = -3$

$$\frac{x}{a} + \frac{y}{b} = 1.$$

$$\frac{x}{2} + \frac{y}{-3} = 1$$

$$\frac{-3x + 2y}{-6} = 1$$

$$-3x + 2y = -6$$

$$3x - 2y - 6 = 0.$$

Q- Find the equation of the line which passes through the point $(3,4)$ and the sum of whose intercepts on the axes is 14.

Sol: let the intercepts made by the line on the x-axis & y-axis be a and $(14-a)$...

the required equation is $\frac{x}{a} + \frac{y}{14-a} = 1.$

since it passes through the point $(3,4)$.

$$\frac{3}{a} + \frac{4}{14-a} = 1 \Rightarrow \frac{3(14-a) + 4a}{a(14-a)} = 1$$

$$42 - 3a + 4a = 14a - a^2$$

$$42 + a = 14a - a^2$$

$$a^2 - 14a + a + 42 = 0$$

$$a^2 - 13a + 42 = 0$$

$$a^2 - 6a - 7a + 42 = 0$$

$$a(a-6) - 7(a-6) = 0$$

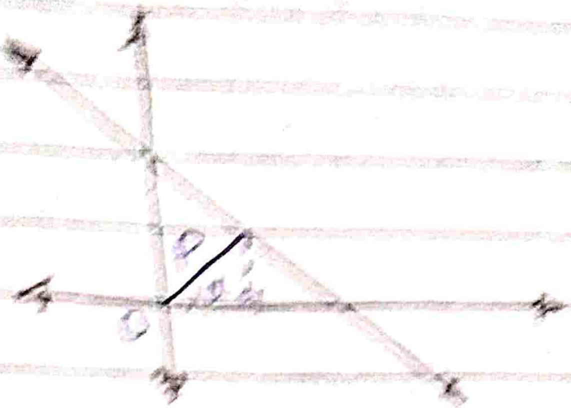
$a = 6$ or $a = 7$.

so, the required equation

$$\frac{x}{6} + \frac{y}{8} = 1$$

$$x + y - 7 = 0.$$

Equation of a line in normal form



$\boxed{x\cos\theta + y\sin\theta = p}$

* Find the equation of a line whose perpendicular distance from the origin is 5 units and the angle between the positive direction of the x-axis and the perpendicular is 30°.

Sol— $p = 5$ and $\theta = 30°$

So, the required solution is $x\cos\theta + y\sin\theta = p$

$x\cos 30° + y\sin 30° = 5$

$x\frac{\sqrt{3}}{2} + y\frac{1}{2} = 5$

$\sqrt{3}x + y = 10$

$\sqrt{3}x + y - 10 = 0$

* General Equation of a line — $\boxed{y = mx + c}$

* REDUCTION OF GENERAL FORM TO STANDARD FORM —

• Given equation of a line $Ax + By + C = 0$

I Slope Intercept form –

$Ax + By + C = 0 \Rightarrow By = -Ax - C$

$\Rightarrow y = \frac{-Ax - C}{B}$

$y = mx + c$

$$\Rightarrow \boxed{y = \left(\frac{-A}{B}\right)x + \left(\frac{-C}{B}\right)}$$

Where,

$m = \frac{-A}{B}$, $c = \frac{-C}{B}$

II Intercepts form

$Ax + By + C = 0 \Rightarrow Ax + By = -C$

$\Rightarrow \left(\frac{A}{-C}\right)x + \left(\frac{B}{-C}\right)y = 1$

$\frac{x}{a} + \frac{y}{b} = 1$

$$\Rightarrow \boxed{\frac{x}{\left(\frac{-C}{A}\right)} + \frac{y}{\left(\frac{-C}{B}\right)} = 1}$$

Where, $a = \frac{-C}{A}$, $b = \frac{-C}{B}$.

III Normal form

$Ax + By + C = 0 \Rightarrow \frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y + \frac{C}{\sqrt{A^2+B^2}}$

$x\cos a + y\sin a = p$

$$\boxed{\frac{-A}{\sqrt{A^2+B^2}}x + \frac{-B}{\sqrt{A^2+B^2}} = \frac{C}{\sqrt{A^2+B^2}}}$$

Where, $\cos a = \frac{-A}{\sqrt{A^2+B^2}}$; $\sin a = \frac{-B}{\sqrt{A^2+B^2}}$; $p = \frac{C}{\sqrt{A^2+B^2}}$

Q. Reduce the equation $\sqrt{3}x + y + 2 = 0$.
(i) slope-intercepts form. find slope & y-intercept
(ii) intercepts form and find the intercepts on the axes

Ans:- $\sqrt{3}x + y + 2 = 0$
(i) $y = -\sqrt{3}x - 2$ $m = -\sqrt{3}$ & $c = -2$.

(ii) $\sqrt{3}x + y + 2 = 0$
$\sqrt{3}x + y = -2$ x-intercept   y-intercept
$\frac{\sqrt{3}x}{-2} + \frac{y}{-2} = 1$ $a = \frac{-2}{\sqrt{3}}$, $b = -2$.

$\therefore \frac{x}{\left(\frac{-2}{\sqrt{3}}\right)} + \frac{y}{(-2)} = 1$

Q. Reduce the equation $x + \sqrt{3}y + 5 = 0$ to the normal form.

Ans:- $x + \sqrt{3}y + 5 = 0$
$-x - \sqrt{3}y = 5$ [keeping constant +ve].

Now,
On dividing by $\sqrt{A^2 + B^2} = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$.

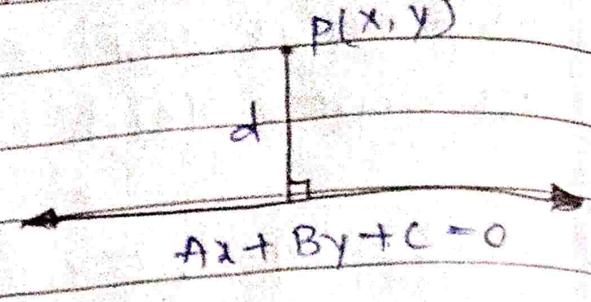
Again,
$-\frac{x}{2} - \frac{\sqrt{3}y}{2} = \frac{5}{2}$

$\cos a = -\frac{1}{2}$; $\sin a = \frac{-\sqrt{3}}{2}$; $p = \frac{5}{2}$

$a = 120°$ $\tan a = \frac{\sin a}{\cos a} = \frac{\frac{-\sqrt{3}}{2}}{\frac{-1}{2}} = \sqrt{3} = 30°$
$\downarrow$
$240°$

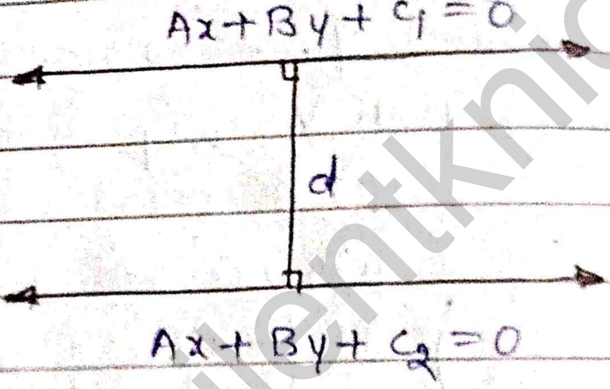
Distance of a point from a line.

$$\boxed{d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}}$$



Distance between two parallel lines

$$\boxed{d = \frac{|C_2 - C_1|}{\sqrt{A^2 + B^2}}}$$



Distance between two parallel lines $y = mx + c_1$ and $y = mx + c_2$ is given by

$$\boxed{d = \frac{|c_2 - c_1|}{\sqrt{1 + m^2}}}$$

Q. find the distance of the point (4, 1) from the line $3x - 4y + 12 = 0$.

$$d = \frac{|3 \times 4 + (-4 \times 1) + 12|}{\sqrt{3^2 + (-4)^2}}$$

$$= \frac{|12 - 4 + 12|}{5} = \frac{20}{5} = 4 \text{ units}$$

Q. find the distance between the parallel lines $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$.

Ans. $15x + 8y - 34 = 0$

$$y = -\frac{15}{8}x + \frac{17}{4} \quad \text{—— (I)} \quad m = -\frac{15}{8},\ c = \frac{17}{4}$$

$15x + 8y + 31 = 0$

$$y = -\frac{15}{8}x - \frac{31}{8} \quad \text{—— (II)} \quad m = -\frac{15}{8},\ c = -\frac{31}{8}$$

$$\text{distance } (d) = \frac{|c_2 - c_1|}{\sqrt{1+m^2}}$$

$$= \frac{\left|-\frac{31}{8} - \frac{17}{4}\right|}{\sqrt{1+\left(\frac{-15}{8}\right)^2}} = \frac{\left|\frac{-31-34}{\cancel{8}}\right|}{\frac{\sqrt{8^2+(-15)^2}}{\cancel{8}}}$$

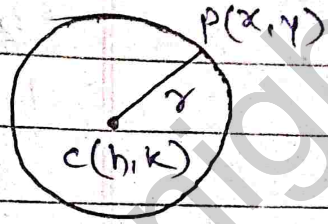
$$d = \left|\frac{-65}{17}\right| \quad ; \quad d = \frac{65}{17} \text{ units.}$$

Circle

circle – The set of all points in a plane which are at a constant distance from a fixed point in the plane.

· Equation of a circle in standard form -

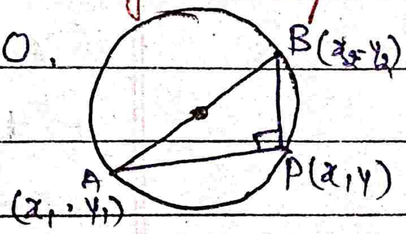
$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$



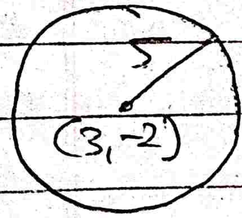
· Equation of a circle with centre at the origin and radius r is given by $x^2+y^2=r^2$.

· Equation of a circle with $A(x_1, y_1)$ and $B(x_2, y_2)$ as the end points of a diameter is given by

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0.$$



· Find the equation of a circle with centre (3, −2) & radius 5.



Equation of a circle -

$$(x-x_1)^2 + (y-y_1)^2 = r^2$$

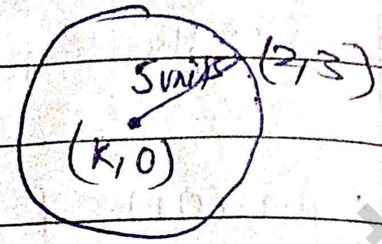
$$(x-3)^2 + [y-(-2)]^2 = 5^2$$

$$(x-3)^2 + (y+2)^2 = 5^2$$

$$x^2+9-6x+y^2+4+4y = 25$$

$$x^2-6x+4y-12=0.$$

Q. find the eqⁿ of a circle of radius 5 units, whose centre lies on the x-axis and which passes through the point (2, 3).



Sol^n: $r = 5$ units.

So, the required solution -

$(2-K)^2 + (3-0)^2 = 5^2$

$4 + K^2 - 4K + 9 = 25$

$K^2 - 4K + 13 - 25 = 0$

$K^2 - 4K - 12 = 0$

$K^2 - 6K + 2K - 12 = 0$

$K(K-6) + 2(K-6) = 0$

$(K-6)(K+2) = 0$ $\quad$ $K = 6$ or $K = -2$.

Hence, the required equation of the circle.

$(x-6)^2 + (y-0)^2 = 5^2$ / $(x+2)^2 + (y-0)^2 = 5^2$

$x^2 + y^2 - 12x + 11 = 0$ / $x^2 + y^2 + 4x - 21 = 0$.

Q. Find the equation of a circle, the end points of one of whose diameters are A(2, -3) & B(-3, 5).

$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$

$(x-2)(x+3) + (y+3)(y-5) = 0$

$x^2 + y^2 + x - 2y - 21 = 0$.

General Equation of a circle —

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

its centre is $(-g, -f)$ and radius $= \sqrt{g^2+f^2-c}$.

Q) Show that the eqn $x^2+y^2-6x+4y-36=0$ represents a circle. Also, find its centre and radius.

Sol:- $x^2+y^2-6x+4y-36=0$ it is the form of
$x^2+y^2+2gx+2fy+c=0$.

where $2g=-6$, $2f=4$, $c=-36$.
$g=-3$, $f=2$, $c=-36$.

centre $(-g,-f) = (3,-2)$.
radius $= \sqrt{g^2+f^2-c} = \sqrt{(-3)^2+(2)^2-36}$
$= \sqrt{9+4+36}$
$= \sqrt{13+36} = \sqrt{49} = 7$ units.

Q) Find the equation of a circle passing through the points $(5,7)$, $(6,6)$ and $(2,-2)$. Find its centre and radius.

Sol:- let the required equation of the circle be
$x^2+y^2+2gx+2fy+c=0$.

Since, it ~~passes~~ through each points $(5,7)$ $(6,6)$ and $(2,-2)$ —

$$x^2+y^2+2gx+2fy+c=0.$$

(5, 7) $\quad 5^2+7^2+2g\times 5+2f\times 7+c=0$

$$25+49+10g+14f+c=0$$

$$74+10g+14f+c=0. \quad \text{---} (i)$$

(6, 6) $\quad (6)^2+6^2+2g\times 6+2f\times 6+c=0$

$$36+36+12g+12f+c=0$$

$$72+12g+12f+c=0 \quad \text{---} (ii)$$

(2, -2) $\quad 2^2+(-2)^2+2g\times 2+2f\times -2+c=0$

$$4+4+4g-4f+c=0$$

$$8+4g-4f+c=0 \quad \text{---} (iii)$$

On subtracting eqn (ii) & eqn (i) ; we get.

$$72+12g+12f+\cancel{c}-74-10g-14f-\cancel{c}=0$$

$$-2+2g-2f=0$$

$$2g-2f=2$$

$$g-f=1 \quad \text{---} (iv)$$

On subtracting eqn (iii) & eqn (ii) , we get

$$8+4g-4f+\cancel{c}-72-12g-12f-\cancel{c}=0$$

$$-64-8g-16f=0$$

$$8g+16f=-64$$

$$g+2f=-8 \quad \text{---} (v).$$

On solving eqⁿ (iv) & (v); we get.

$$\cancel{g} - f = 1$$
$$\cancel{-g} + 2f = -8$$
$$-3f = 9$$
$$\boxed{f = -3}.$$

putting $f = -3$ in eqⁿ (iv); we get:

$$g - f = 1$$
$$g - (-3) = 1$$
$$g + 3 = 1$$
$$g = 1 - 3$$
$$g = -2.$$

$\therefore$ $g = -2$ & $f = -3$.

Substituent $g = -2$ & $f = -3$ in eqⁿ (iii).

$$8 + 4g - 4f + c = 0$$
$$8 + 4\times -2 - 4\times -3 + c = 0$$
$$\cancel{8} - \cancel{8} + 12 + c = 0$$
$$\boxed{c = -12}.$$

So, the required equation is

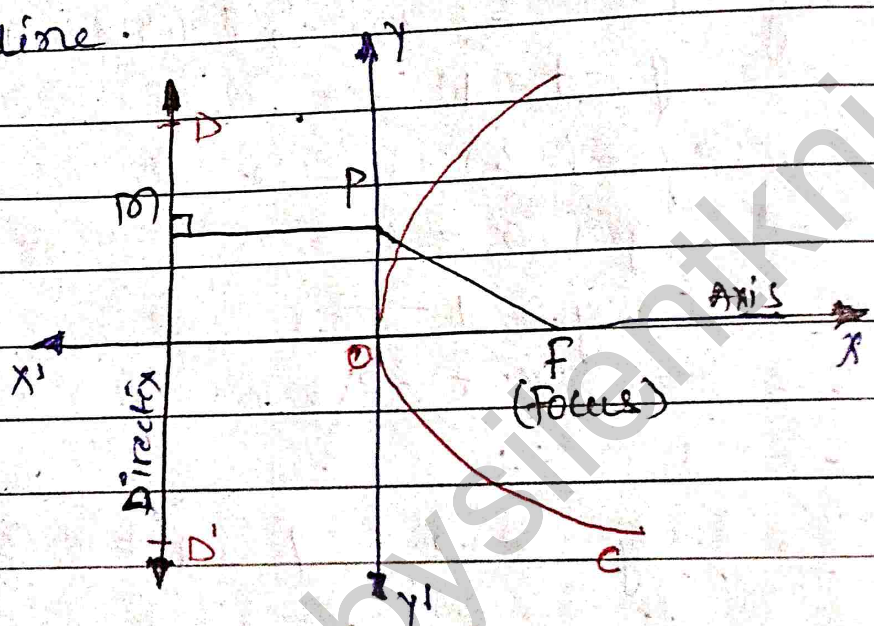
$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$$x^2 + y^2 + \cancel{2gx} 2x\times -2 + 2y\times -3 + 12 = 0$$
$$x^2 + y^2 - 4x - 6y + 12 = 0.$$

centre of circle $= (-g, -f) = (2, 3)$

$\sqrt{g^2 + f^2 - c} = \sqrt{2^2 + 3^2 + 12} = \sqrt{25} = 5$ unit

PARABOLA

It is the path traced by a point which moves in a plane in such a way that its distance from a fixed point is always equal to its distance from a fixed line both lying in the same plane, whereas the given fixed point does not lie on the given line.



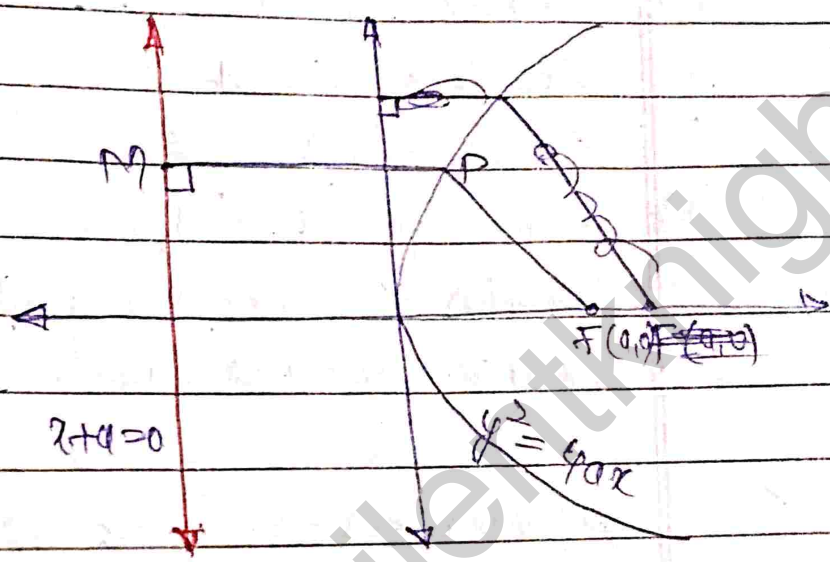
The fixed point is called focus.
The fixed line called its directrix.

Focal chord :- Any chord of a parabola passing through its focus is called a focal chord.

Focal distance of a point :- The distance of any point of a parabola from its focus is called the focal distance of a point.

latus rectum of a parabola – A chord of a parabola, passing through its focus and perpendicular to its axis, is called the lectus rectum of the parabola.

$y^2 = 4ax$, $a > 0$



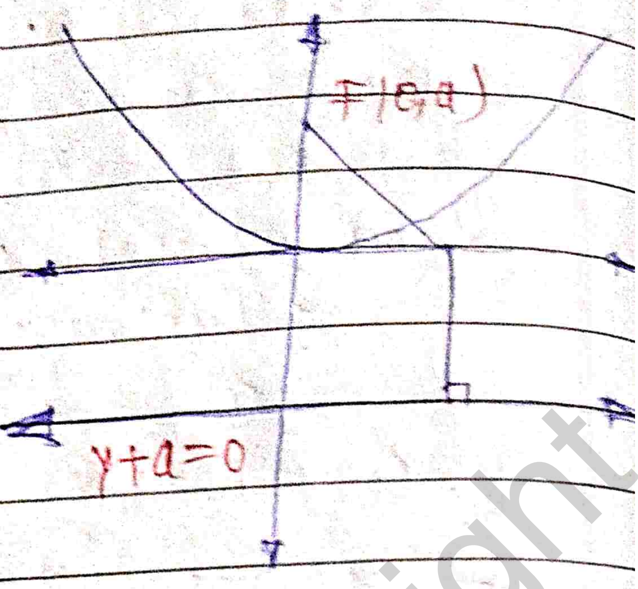
i) Focus is $F(a,0)$
ii) vertex is $O(0,0)$
iii) directix is the line
$x + a = 0$
iv) axis is the line
$y = 0$
v) length of the lactus rectum is $4a$.
vi) lactus rectum is the line $x = a$.

left - handed parabola.

$y^2 = -4ax$, $a > 0$

i) Focus is $F(-a,0)$
ii) vertex is $O(0,0)$
iii) Directix is the line $x - a = 0$
iv) Axis is the line $y = 0$
v) length of the lactus rectum is $4a$.
vi) lactus rectum is the line $x = -a$.

Upword parabola:



$x^2 = 4ay$, $a > 0$

i) $F(0, a)$
ii) vertex is $O(0,0)$
iii) Directix is the line $y + a = 0$.
iv) Axis is the line $x = 0$
v) length of lactus rectum is $4a$.
vi) lactus rectum is the line $y - a = 0$.

Downword parabola:

$x^2 = -4ay$, $a > 0$

i) $F(0, -a)$
ii) $O(0, 0)$
iii) Directix is the line $y - a = 0$
iv) Axis is the line $x = 0$
v) length of lactus rectum is $4a$
vi) lactus rectum is the line $y + a = 0$.

Q. Find the co-ordinates of the focus and vertex, the equations of the directrix and the axis, and length of the lactus rectum of the parabola. $y^2 = 8x$.

$y^2 = 8x$ ∴ $4a = 8$

$y^2 = 4ax$ $\boxed{a = 2}$

$F(a,0)$ i.e $F(2,0)$

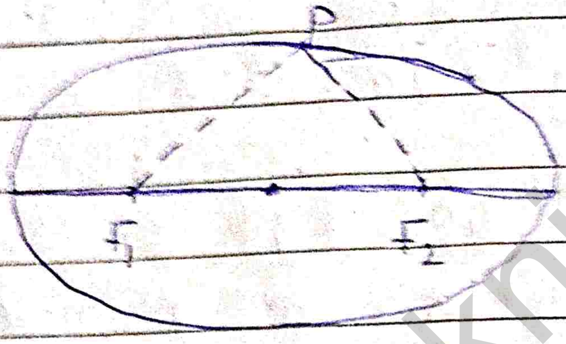
Its vertex is $O(0,0)$

Equation of the directrix is $x = -a$ i.e $x = -2$

length of lactus rectum $= 4a = 4 \times 2 = 8$ units.

Ellipse

Ellipse – It is the path traced by a point which moves in a plane in such a way that the sum of its distance from two fixed points in the plane is a constant.

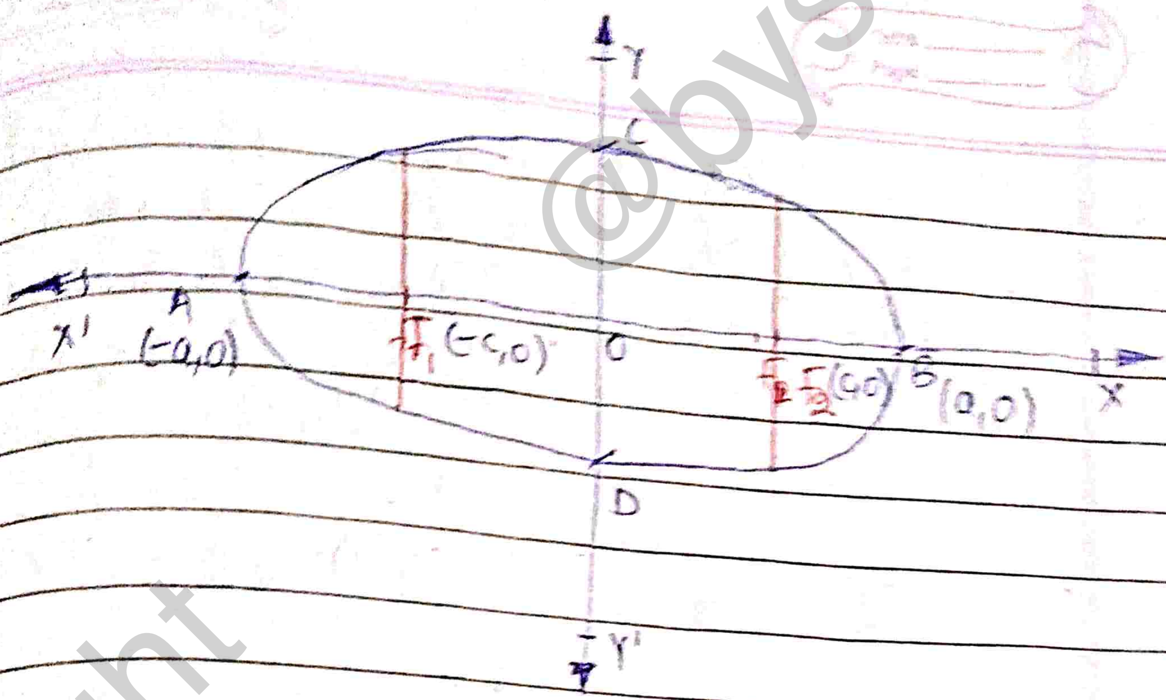


Centre of the ellipse – The midpoint of the line segment joining the foci, is called the centre of the ellipse.

Major axis – The line segment through the foci of the ellipse with its end points on the ellipse, is called its Major axis.

Minor axis – The line segment through the centre and perpendicular to the major axis with its end points on the ellipse. is called its minor Axis.

Vertices of an ellipse – The end points of the major axis of an ellipse are called its vertices.



i) the standard form of equation of a horizontal ellipse is —

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad , \quad 0 < b < a .$$

ii) Its centre is O(0,0)

iii) Its vertices are A(-a, 0) and B(a,0).

iv) Its foci are $F_1(-c,0)$ and $F_2(c,0)$.

where $c^2 = (a^2 - b^2)$.

v) length of the major axis, AB = 2a.

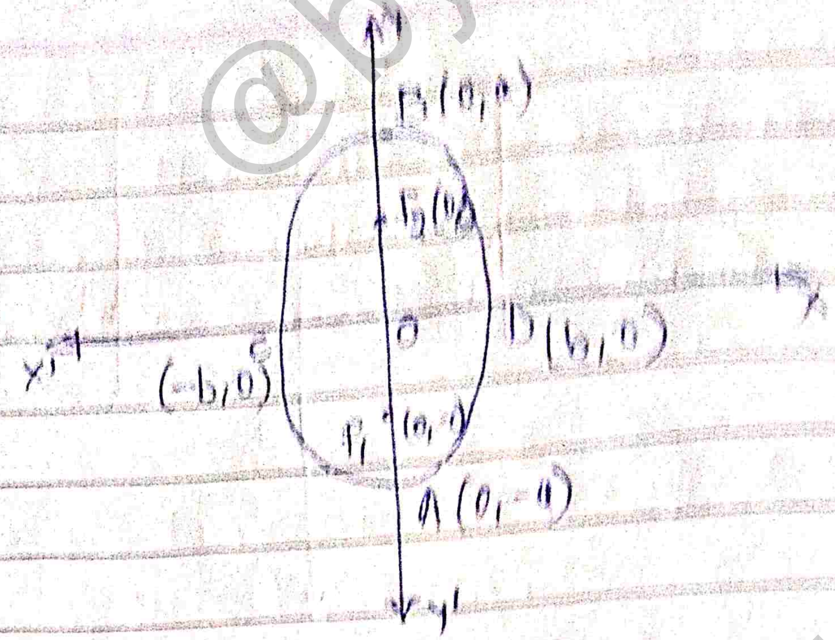
length of the minor axis, CD = 2b

vi) Equation of the major axis is Y = 0

" " minor axis is x = 0

vii) length of the lactus rectum $= \frac{2b^2}{a}$.

viii) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$.



i) Standard form of the Equation of vertical ellipse is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, $a > b$

ii) Its centre is $O(0,0)$

iii) Its vertices $A(0,-a)$ & $B(0,a)$

iv) foci are $f_1(0,-c)$ & $f_2(0,c)$
where $a^2 - b^2 = c^2$

i.e $f_1(0,-ae)$ and $f_2(0,ae)$

v) length of major axis $= 2a$
" minor axis $= 2b$

vi) Equation of the major axis is $x = 0$
" minor axis is $y = 0$

vii) length of lactus rectum $= \frac{2b^2}{a}$

viii) Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$

Find the lengths of the major and minor axes; co-ordinates of the vertices and the foci, the eccentricity and length of the lactus rectum of the ellipse.

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

This is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

So, it is an equation of a horizontal ellipse.

$c = \sqrt{a^2 - b^2} = \sqrt{16-9} = \sqrt{7}.$

length of major axis $2a = 2 \times 4 = 8.$

length of minor axis $2b = 2 \times 3 = 6.$

co-ordinates of the vertices $= O(-4,0)$ & $(4,0)$.

foci $f_1(-\sqrt{7}, 0)$ & $f_2(\sqrt{7}, 0)$.

Eccentricity $(e) = \frac{c}{a} = \frac{\sqrt{7}}{4}.$

length of the lactus rectum $= \frac{2b^2}{a} = \frac{2 \times 9}{4_2} = \frac{9}{2}$ units

Find the Equation of an ellipse whose vertices are at $(\pm 5, 0)$ and foci at $(\pm 4, 0)$.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

$F(\pm 4, 0)$

$c = \pm 4.$

$\frac{x^2}{25} + \frac{y^2}{9} = 1.$

$A(\pm 5, 0)$

$a = \pm 5.$

$c^2 = a^2 - b^2$

$b^2 = a^2 - c^2$
$= (5)^2 - (4)^2 = 3^2.$

Hyperbola

Hyperbola — It is the set of all points in a plane, the difference of whose distance form two fixed points in the plane is a constant.

The two fixed points are called the foci of the hyperbola.

* Centre of the hyperbola — The midpoint of the line segment joining the foci is called the centre of the hyperbola.

Transverse Axis — The line through the foci of the hyperbola is called its transverse axis.

Conjugate axis — The line through the centre and perpendicular to the transverse axis of the hyperbola is called its Conjugate axis.

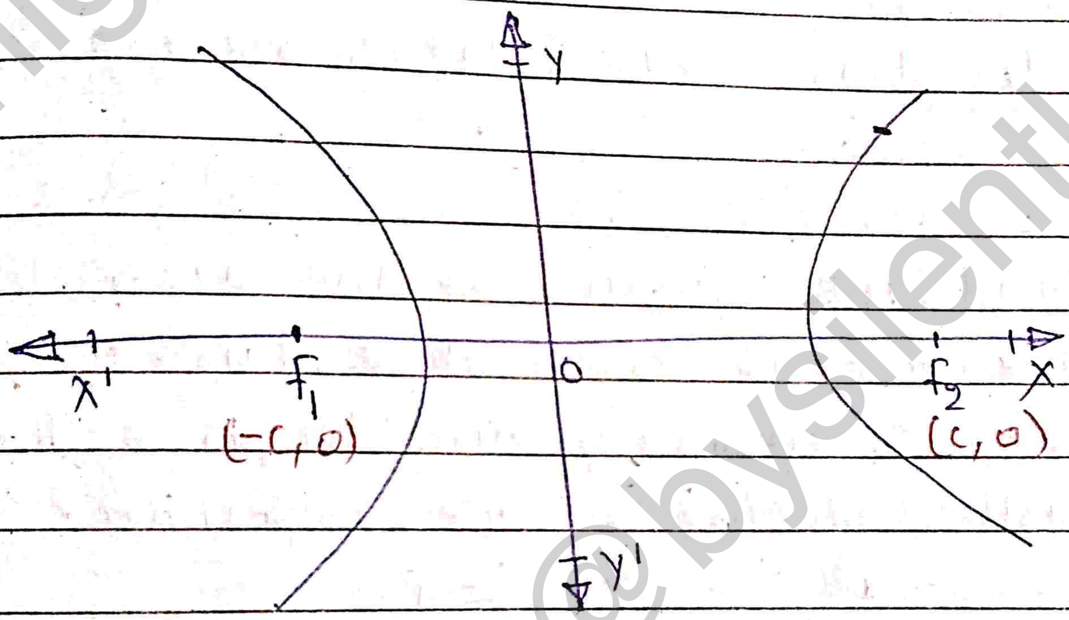
Vertices of the hyperbola — The points at which the hyperbola intersects the transverse axis are called its vertices.

length of transverse axis – The distance between the two vertices of a hyperbola is called the length of its transverse axis.

Eccentricity – The ratio $\frac{c}{a}$ is always constant, called the eccentricity of the hyperbola.

It is denoted by e.

Here, $c > a \Leftrightarrow \frac{c}{a} > 1 \Leftrightarrow e > 1$.



i) standard equation of a horizontal hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

ii) Its centre O(0,0)

iii) X'OX is the transverse axis

YOY' is the conjugate axis

iv) Its foci $F_1(-c,0)$ & $F_2(c,0)$

i.e $F_1(-ae,0)$ & $F_2(ae,0)$.

(v) Its vertices $A(-a,0)$ & $B(a,0)$.

(vi) Its eccentricity $e = \frac{c}{a} = \frac{\sqrt{a^2+b^2}}{a}$.

vii) length of the transverse axis $= 2a$
its equation is $y = 0$

viii) length of the conjugate axis $= 2b$
its equation is $x = 0$.

ix) length of its latus rectum $= \frac{2b^2}{a}$.

Q.) Find the lengths of the axes, the co-ordinates of vertices and the foci; the eccentricity and length of the lactus rectum of the hyperbola:

$$\frac{x^2}{36} - \frac{y^2}{64} = 1.$$

soln:- $\frac{x^2}{36} - \frac{y^2}{64} = 1.$

$a^2 = 36 \quad \therefore \quad a = \pm 6$

$b^2 = 64 . \quad b = \pm 8$

$c = \sqrt{a^2+b^2} = \sqrt{36+64} = \sqrt{100} = 10.$

i) length of transverse axis $= 2a = 2 \times 6 = 12$
" conjugate axis $= 2b = 2 \times 8 = 16$

co-ordinates of vertices $A(-a,0)$ & $B(a,0)$
$A(-6,0)$ & $B(6,0)$.

foci $f_1(-c,0)$ & $f_2(c,0) \Rightarrow f_1(-10,0)$ & $f_2(10,0)$.

eccentricity $(e) = \frac{c}{a} = \frac{10}{6} = \frac{5}{3}$.

length of the lectus rectum $= \frac{2b^2}{a} = \frac{2 \times 64}{\cancel{6}3} = \frac{64}{3}$.

J.M.S civil Engineering

MATHEMATICS

Algebra

UNIT-3

I. complex Number.
II. partial fraction.
III. permutation and combination
IV. Binomial theorem
V. Matrices and Determinant.

I. complex Number

Imaginary Numbers:- if the square of a given number is negative then such a number is called an imaginary Number.

For example, $\sqrt{-1}$, $\sqrt{-2}$ etc are Img. No.

$\sqrt{-1}$ by Greek letter iota 'i'

Thus,

$$\sqrt{-4} = 2i, \quad \sqrt{-9} = 3i, \quad \sqrt{-5} = i\sqrt{5} \text{ etc}$$

powers of i.

* $i^0 = 1$
* $i^1 = i$
* $i^2 = -1$
* $i^3 = -i$
* $i^4 = 1$

Ex:- $i^{23} = i^{(4\times5)+3} = (i^4)^5 \times i^3 = i^3 = -i$.

Ex:- $i^{998} = i^{4\times249+2} = (i^4)^{249} \times i^2 = -1$.

Ex:- $i^{-998} = \frac{1}{i^{998}} \times \frac{i^2}{i^2} = \frac{i^2}{i^{1000}} = \frac{-1}{1} = -1$.

Ex:- $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$.

$i^n(1 + i^1 + i^2 + i^3) = 0$

$i^n(1 + i - 1 - i) = 0$.

Ex:- $(1+i)^4 \times \left(1 + \frac{1}{i}\right)^4 = 16$.

$(1+i)^4 \left(1 + \frac{i}{i^2}\right)^4$

$(1+i)^4 (1-i)^4 = \left[(1^2 - (i)^2)\right]^4$

$= [1+1]^4 = 2^4 = 16$.

Ex:- $\sqrt{-25} \times \sqrt{-49} \Rightarrow 5i \cdot 7i = 35i^2 = -35$.

Complex Numbers — The numbers of the form $(a+ib)$, where a and b are real numbers and $i = \sqrt{-1}$, are known as complex numbers.

It is denoted by C.

i.e. $C = \{(a+ib) : a, b \in R\}$.

Ex:- $(5+8i)$, $(-3+\sqrt{2}i)$, $\left(\frac{2}{3} - \frac{5}{7}i\right)$ ... is a complex.

* Conjugate of a complex Number.

Conjugate of a complex number $z = (a + ib)$ is defined as, $\bar{z} = (a - ib)$

(i) $\overline{(3+8i)} = (3-8i)$

(ii) $\overline{(-6-2i)} = (-6+2i)$.

(iii) $\overline{-3} = -3$.

* Modulus of a complex Number

$z = (a+ib)$ then, $|z| = \sqrt{a^2+b^2}$

Ex: (i) $z = 2+3i$; $|z| = \sqrt{2^2+3^2} = \sqrt{13}$

(ii) $z = -5-4i$; $|z| = \sqrt{(-5)^2+(-4)^2} = \sqrt{41}$.

* Equality of complex numbers.

$z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$.

Ex:- $2y + (3x-y)i = 5-2i$.

$2y = 5$; $3x - y = -2$

$\boxed{y = \frac{5}{2}}$

$3x - \frac{5}{2} = -2$

$3x = -2 + \frac{5}{2} = \frac{1}{2}$

$x = \frac{1}{2} \times \frac{1}{3}$

$\boxed{x = \frac{1}{6}}$

Sum and difference of complex numbers.

$z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$.

(i) $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$.

(ii) $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$.

(i) closure property.

$z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$.

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$$
$$= (a_1 + a_2) + i(b_1 + b_2).$$

(ii) Commutative law $\quad z_1 + z_2 = z_2 + z_1$

(iii) Associative law $\quad (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(iv) Exitence of Additive Identity

$$z + 0 = 0 + z = z.$$

(v) Existence of Additive Inverse.

$$z + (-z) = (-z) + z = 0.$$

Multiplication of complex number.

$z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$$
$$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$$

$\therefore z_1 z_2 = Re(z_1) \cdot Re(z_2) - Im(z_1) \cdot Im(z_2)$.

(i) $z_1 = (3+2i)$ & $z_2 = (5+4i)$.

$$z_1 z_2 = (3+2i)(5+4i)$$
$$= (3\times5 + 3\times4i + 2i\times5 + 2i\times4i)$$
$$= (15 + 12i + 10i - 8)$$
$$= 7 + 22i$$

I Closure property

$$z_1 z_2 = (a_1+ib_1)(a_2+ib_2).$$

II Commutative law: $z_1 z_2 = z_2 z_1$

III Associative law: $(z_1 z_2) z_3 = z_1(z_2 z_3)$

IV Existence of multiplicative Identity

$$z\times1 = 1\times z = z.$$

V Existence of multiplicative Inverse.

$$z = (a+ib)$$

$$z^{-1} = \frac{1}{z} = \frac{1}{(a+ib)}$$

$$\Rightarrow \frac{1}{(a+ib)} \cdot \frac{(a-ib)}{(a-ib)} = \frac{(a-ib)}{a^2+(b^2)}$$

$$z\times z^{-1} = z^{-1}\times z = 1.$$

$$\boxed{z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2} \qquad \because z\bar{z} = |z|^2}$$

Division of two complex numbers:-

let z_1 and z_2 be complex numbers such that $z_2 \neq 0$.

Then, $\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = z_1 z_2^{-1}$.

Ex. $z_1 = 6+3i$ and $z_2 = 3-i$

$$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1}$$

$$= (6+3i)\left(\frac{1}{3-i}\right)^{-1}$$

$$= (6+3i)\left(\frac{3+i}{10}\right)$$

$$= \frac{(18-3)+i(15)}{10}$$

$$= \frac{15+15i}{10}$$

$$= \frac{3+3i}{2} = \frac{3(1+i)}{2}.$$

$$z_2^{-1} = \frac{\overline{z_2}}{|z_2|^2}$$

$$= \frac{\overline{(3-i)}}{|3-i|^2}$$

$$= \frac{3+i}{3^2+(-1)^2}$$

$$= \frac{3+i}{10}.$$

polar Representation of complex No.

polar form of $z = x + iy$ is $r(\cos\theta + i\sin\theta)$

(i) $r = |z| = \sqrt{x^2 + y^2}$.

(ii) $\tan a = \left|\frac{y}{x}\right| = \left|\frac{Im(z)}{Re(z)}\right|$

(iii) When $-\pi < \theta \leq \pi$, then θ is the principal argument of z.

Quad. in which z lies	arg(z)
I	$\theta = a$
II	$\theta = \pi - a$
III	$\theta = -(\pi - a)$
IV	$\theta = -a$ or $(2\pi - a)$

Example:— convert in polar form.

(i) 3.

$z = 3 + 0i$

Let it polar form be $z = r(\cos\theta + i\sin\theta)$

$\tan a = \left|\frac{Im(z)}{Re(z)}\right| = \frac{0}{3} = 0.$

$a = 0^\circ$.

Now, $z = r(\cos 0^\circ + i\sin 0^\circ)$

clearly, $z = 3 + 0i$ is represented by the point $P(3, 0)$, which lies on the positive side of the x-axis.

Example : Argument of the complex No.

(i) $1+i$

$z = 1+i$

$Re(z) = 1$ & $Im(z) = 1$

$\tan\alpha = \left|\frac{Im(z)}{Re(z)}\right| = \frac{1}{1} = 1$

$\alpha = \frac{\pi}{4}$.

the point representing $z = 1+i$ is $P(1,1)$, which lies in the first quadrant.

Example :- convert $(1+i\sqrt{3})$ into polar form.

$z = 1+\sqrt{3}i$

$|z| = r = \sqrt{1^2+(\sqrt{3})^2}$

$= \sqrt{1+3} = \sqrt{4} = 2$.

let ~~it~~ ~~if~~ the polar form is $r(\cos\theta + i\sin\theta)$.

$z = 1+\sqrt{3}i$; $Re(z) = 1$ & $Im(z) = \sqrt{3}$

$\tan\alpha = \frac{Im(z)}{Re(z)} = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\alpha = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$. $\boxed{\therefore \alpha = \frac{\pi}{3}}$.

clearly, it is the point is representing $P(1,\sqrt{3})$ in the first quadrant.

The required polar form of $z = 1+\sqrt{3}i$ is $r(\cos\theta + i\sin\theta)$

$2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ Ans

Convert $4(\cos 300^\circ + i\sin 300^\circ)$ into Cartesian form.

$4(\cos 300^\circ + i\sin 300^\circ)$

$4(\cos(360-60) + i\sin(360-60))$

$4[\cos 60^\circ + i\sin 60^\circ]$

$4\left[\frac{1}{2} + \frac{i\sqrt{3}}{2}\right] = 2 - 2i\sqrt{3}$.

Quadratic Equations (with complex roots)

A polynomial equation of degree n has at the most n roots.

Ex:- solve $x^2+3=0$

$x^2+3=0$

$x^2=-3$

$x=\sqrt{-3} = \pm\sqrt{3}i$

$x = +\sqrt{3}i, -\sqrt{3}i$.

$x^2+3x+9=0$.

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{-3 \pm \sqrt{(+3)^2 - 4\times1\times9}}{2\times1}$

$= -3 \pm \sqrt{9-36}/2$

$= \frac{-3 \pm \sqrt{-27}}{2}$

$= \frac{-3 \pm 3\sqrt{3}i}{2}, \frac{-3}{2} + \frac{3\sqrt{3}i}{2}, \frac{-3}{2} - \frac{3\sqrt{3}i}{2}$

Square roots of a complex No.

$\sqrt{a+ib}$

let $\sqrt{a+ib} = x+iy$

On squaring both sides -

$a+ib = (x+iy)^2$

$a+ib = x^2+(iy)^2+2x(iy)$

$a+ib = x^2-y^2+i2xy.$

$a = x^2-y^2$ & $b = 2xy.$

$\Rightarrow (x^2+y^2) = \sqrt{(x^2-y^2)^2+4x^2y^2} = \sqrt{a^2+b^2}$

Q1 - $\sqrt{6+8i}$

let $\sqrt{6+8i} = x+iy$

On squaring both sides of (i), we get

$6+8i = (x+iy)^2$

$6+8i = x^2-y^2+2xyi$

$6 = x^2-y^2$ & $8 = 2xy$

$6 = x^2-y^2$ & $xy = 4.$

$\Rightarrow (x^2+y^2) = \sqrt{(x^2-y^2)^2+4x^2y^2}$

$= \sqrt{6^2+4\times16}$

$= \sqrt{36+64}$

$= \sqrt{100} = 10.$

$x^2-y^2 = 6$ & $x^2+y^2 = 10$

$2x^2 = 16$ / $y^2 = 2.$ $x = \pm2\sqrt{2}$

$x^2 = 8$ $y = \pm\sqrt{2}$

II. partial fraction

if $f(x)$ and $g(x)$ are polynomials, then the quotient $\frac{f(x)}{g(x)}$ is termed as rational algebraic fraction.

$$\frac{3x}{x^2-16}, \quad \frac{x^2+5x}{x^2+2x-8}, \quad \frac{x^3-2x^2+1}{x^2-1} \text{ etc}$$

- proper fraction – In any fraction if the numerator is lower degree than its denominator, it is called proper fraction.

Ex!- $\frac{2x+5}{x^2+1}, \frac{3x+8}{x^2+4x+3}$.

- Improper fraction – In Improper fraction the degree of numerator is equal to or greater than the degree of its denominator.

Ex!- $\frac{x^2+2x}{x^2+3x-5}$ and $\frac{x^3-2x^2+1}{2x^2+2x-1}$, partial fraction - if we split up the given algebraic fraction into different fractions whose denominators are the factors of the denominator of the given fraction, these fractions are called partial fractions.

Ex! - $\frac{7x-1}{2x^2-x-1} = \frac{3}{2x+1} + \frac{2}{x-1}$.

Resolving into partial fractions —

When the factors of the denominator are linear and non-repeating.

When one or more linear factors of the denominator are repeating.

When the denominator has one or more quadratic non-repeating factors.

When the denominator has one or more quadratic repeating factors.

Case - I.
When Denominator has linear non repeated factors:.

$$\frac{1}{(x+4)(x+6)} = \frac{A}{x+4} + \frac{B}{x+6}.$$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}.$$

$$\frac{x^2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

$$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = 1 + \frac{A}{x-4} + \frac{B}{x-5} + \frac{C}{x-6}.$$

$$\frac{2x^3+7x^2-2x-2}{2x^2+x-6} = \frac{A}{x+2} + \frac{B}{2x-3} + (x+1)$$

Case-II When the factors in the denominator are linear and repeated.

Ex!- $$\frac{9}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$$\frac{x^2+x}{(x-1)^3} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}.$$

(ii) When the Denominator contains one or more factors which are non-repeated quadrate.

$$\frac{3x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\frac{42-19x}{(x^2+1)(x-4)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{x-4}.$$

$$\frac{x^4}{x^3+1} = x - \frac{x}{(x+1)(x^2-x+1)}$$

$$= x - \left(\frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}\right).$$

$$\frac{x^2+6}{(x^2+3)(x^2+5)} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{x^2+5}.$$

(iii) When Denominator has repeated Quadratic factor

$$\frac{1}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$\frac{3x^2+5x+3}{(x^2+x+1)^2} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2}$$

III. permutation and combination.

Factorial - The factorial of a positive integer n is defined as the product of all positive integers less than or equal to n.

It is denoted by $n!$.

$$n! = n(n-1)(n-2)(n-3)\ldots\ldots 3\times2\times1.$$

$$n! = n(n-1)!$$

$$\boxed{(n-1)! = \frac{n!}{n}}$$

put $n=1$ $\quad (1-1)! = \frac{1!}{1}$

$$\boxed{0! = 1}$$

put $n=0$ $\quad (0-1)! = \frac{0!}{0}$

$$(-1)! = \frac{1}{0} \quad \text{undefined.}$$

put $n=\frac{1}{2}$ $\quad \left(\frac{1}{2}-1\right)! = \frac{\frac{1}{2}!}{1/2}$

$$\left(-\frac{1}{2}\right)! = \frac{1/2\,!}{1/2} \quad \text{undefined.}$$

- Factorial is not defined for proper fraction or neg permutations– A permutation is an arrangement of a number of objects in a definite order taken some or all at a time.

$$\boxed{{}^{n}P_{r} = \frac{n!}{(n-r)!}}$$

Ex:–

$${}^{10}P_{6} = \frac{10!}{(10-6)!} = \frac{10!}{4!} = \frac{10\times9\times8\times7\times6\times5\times\cancel{4!}}{\cancel{4!}} = 151200.$$

$${}^{n}P_{n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

$${}^{n}P_{0} = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1.$$

– : Combination : –

A Combination is an unordered collection of some or all of the objects in a set.

Ex:–

$$\boxed{{}^{n}C_{r} = \frac{n!}{r!\,(n-r)!}}$$

$${}^{n}C_{r} = \frac{{}^{n}P_{r}}{r!}$$

$$\Rightarrow \boxed{{}^{n}P_{r} = {}^{n}C_{r}\cdot r!}$$

* $^{20}C_{18} = \frac{20!}{18!(20-18)!} = \frac{\cancel{20}^{10} \times 19 \times \cancel{18!}}{\cancel{18!} \times \cancel{2!}} = 190.$

* $^nP_4 = 30\,^nC_5 \Rightarrow \frac{\cancel{n!}}{(n-4)!} = 30\frac{\cancel{n!}}{5!(n-5)!}$

$\frac{1}{\cancel{(n-4)!}} = \frac{30}{5!\,(n-5)\cancel{(n-4)!}}$

$5!\,(n-5) = 30$

$\cancel{5 \times 4 \times 3 \times 2 \times 1 \times (n-5) = 30}$

$4n - 20 = 0$

$4n = 20$

$\boxed{n = 5}$

$\frac{1}{(n-4)\cancel{(n-5)!}} = \frac{30}{5!\,\cancel{(n-5)!}}$

$\frac{\cancel{5} \times 4 \times 3 \times \cancel{2} \times 1}{(n-4)} = \cancel{30}$

$n-4 = 4$ $\boxed{n = 8}$.

$^{n+1}C_3 = 2\,^nC_2$

$\frac{(n+1)!}{3!\,(n+1-3)!} = 2\frac{\cancel{n!}}{2!\,(n-2)!}$

$\frac{(n+1)\cancel{n!}}{3!\,\cancel{(n-2)!}} = 2\frac{\cancel{n!}}{2!\,\cancel{(n-2)!}}$

$\frac{n+1}{3 \times \cancel{2!}} = \frac{2}{\cancel{2!}}$

$n+1 = 6$

$\boxed{n = 5}$

BINOMIAL THEOREM

$(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \dots +$
${}^nC_r x^{n-r} a^r + \dots + {}^nC_n a^n ..$

$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n$

$(a+b) =$	$a + b$	1 1
$(a+b)^2 =$	$a^2 + 2ab + b^2$	1 2 1
$(a+b)^3 =$	$a^3 + 3a^2b + 3ab^2 + b^3$	1 3 3 1
$(a+b)^4 =$	$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$	1 4 6 4 1
$(a+b)^5 =$	$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$	1 5 10 10 5 1

Ex! — $(1-2x)^5$

$1 + 5\,1^4(-2x) + 10(1)^3(-2x)^2 + 10(1)^2(-2x)^3 + 5(1)(-2x)^4$
$+ (-2x)^5.$

$\Rightarrow 1 - 10x + 40x^2 - 80x^3 + 80x^4 + 32x^5.$

Ex! — $(\sqrt3 + \sqrt2)^4 - (\sqrt3 - \sqrt2)^4$

$\Rightarrow (\sqrt3)^4 + 4(\sqrt3)^3(\sqrt2) + 6(\sqrt3)^2(\sqrt2)^2 + 4(\sqrt3)(\sqrt2)^3 + (\sqrt2)^4 -$
$[(\sqrt3)^4 + 4(\sqrt3)^3(-\sqrt2) + 6(\sqrt3)^2(-\sqrt2)^2 + 4(\sqrt3)(-\sqrt2)^3$
$+ (\sqrt2)^4]$

$\Rightarrow \cancel{(\sqrt3)^4} + 4(\sqrt3)^3(\sqrt2) + \cancel{6(\sqrt3)^2(\sqrt2)^2} + 4\sqrt3(\sqrt2)^3 + \cancel{(\sqrt2)^4} - \cancel{(\sqrt3)^4} + 4(\sqrt3)^3(\sqrt2)$
$\cancel{- 6(\sqrt3)^2(\sqrt2)^2} + 4(\sqrt3)(\sqrt2)^3 - \cancel{(\sqrt2)^4}$

$12\sqrt6 + 12\sqrt6 + 8\sqrt6 + 8\sqrt6$
$= 24\sqrt6 + 16\sqrt6$

$\sqrt6(40) = 40\sqrt6.$

General term —

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r.$$

$$= {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + \dots + {}^nC_{n-1} x y^{n-1} + {}^nC_n x^0 y^n$$

1st term $= {}^nC_0\, x^n y^0$

2nd term $= {}^nC_1\, x^{n-1} y^1$

- - - - . - - - -

- - - - . - - - -

nth term $= {}^nC_{n-1}\, x^1 y^{n-1}$

$(n+1)$th term $= {}^nC_n\, x^0 y^n$.

So, the $(r+1)$th term (or general term) of the binomial expansion of $(x+y)^n$ ...

$$\boxed{T_{r+1} = {}^nC_r\, x^{n-r} y^r.}$$

* Middle term of the Binomial Expansion

$$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + \dots + {}^nC_{n-1} x^1 y^{n-1} + {}^nC_n x^0 y^n$$

→ The number of terms in the binomial expansion of $(x+y)^n = n+1$.

Case-I : n is even (i.e $n+1$ is odd)

middle term $= \left(\frac{n}{2}+1\right)^{th}$

Case-II : n is odd.

middle term $= \frac{1}{2}(n+1)^{th}$ term & $\left\{\frac{(n+1)}{2}+1\right\}^{th}$ term

p^{th} term from the end is $(a+b)^n$.

$= (n+1-p+1)$ th term from the begning

$= (n-p+2)$ th term from the begning.

Ex! → middle term in the Expansion of $\left(\frac{x}{3}+9y\right)^{10}$

sol^n! → $n = 10$, even

middle term $= \left(\frac{n}{2}+1\right)^{th}$

$= \left(\frac{10}{2}+1\right)^{th} = 6^{th}$ term

$T_{6th} = T_{r+1} = {}^nC_r\, x^{n-r} y^r$

$T_6 = T_{5+1} = {}^{10}C_5 \left(\frac{x}{3}\right)^{10-5} (9y)^5$

$= \frac{10!}{5!(10-5)!}\left(\frac{x}{3}\right)^5 (9y)^5$

$= \frac{10!}{5!.5!}\left(\frac{x^5}{3^5}\right)(9^5)(y^5)$

$= 61236\, x^5 y^5$

Ex:- 5^{th} term from the end $= \left(\frac{x^3}{2} - \frac{2}{x^2}\right)^9$.

Sol^n:- p^{th} term from the end $= (n-p+2)^{th}$ term from the begining.

5^{th} term from the end $= (9-5+2)^{th} = 6$

$$T_6 = T_{5+1} = {}^9C_5 \left(\frac{x^3}{2}\right)^{9-5} \left(\frac{-2}{x^2}\right)^5$$

$$= \frac{9!}{5!(9-5)!} \times \left(\frac{x^3}{2}\right)^4 \times \left(\frac{-2}{x^2}\right)^5$$

$$= \frac{9!}{5! \times 4!} \times \frac{\cancel{x^{12}}}{2^4} \times \frac{(-2)^5}{\cancel{x^{10}}}$$

$$= \frac{9 \times 8 \times 7 \times 6 \times 5!}{5! \times 4 \times 3 \times 2} \times \frac{x^2 (-2^5)}{2^4}$$

$$= -9 \times 7 \times 2 \times 2 \; x^2$$
$$= -252x^2.$$

Ex:- $\left(2x - \frac{1}{x}\right)^{10}$ is independent of x.

Sol^n:- Let T_{r+1} be independent of x.

$$T_{r+1} = (-1)^r \, {}^{10}C_r \, (2x)^{10-r} \left(\frac{1}{x}\right)^r$$

$(-1)^r \times {}^{10}C_r \times (2)^{10-r} \times x^{10-r} \cdot x^{-r}$

$(-1)^r \times {}^{10}C_r \times 2^{10-r} \cdot x^{10-2r}$

if binomial theorem of Expansion is independent of x.

$x^{10-2r} = x^0$

$10 - 2r = 0$

$2r = 10$

$\boxed{r = 5}$

$\Rightarrow (r+1) = (5+1) = 6$

$T_6 = T_{5+1} = (-1)^5 \times {}^{10}C_5 \times (2)^{10-5} \times x^0$

$= -1 \times \frac{10!}{5! \times 5!} \times 2^5$

$= -1 \times \frac{\cancel{10} \times \cancel{9}^{3} \times \cancel{8}^{2} \times 7 \times 6 \times \cancel{5!}}{\cancel{5!} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1} \times 2^5$

$= -3 \times 2 \times 7 \times 6 \times 2^5 = -8064.$

Ex! - $(99)^5$

$(100-1)^5$

$\Rightarrow (100)^5 + 5(100)^4(-1) + 10(100)^3(-1)^2 + 10(100)^2(-1$
$+ 5(100)(-1)^4 + (-1)^5.$

$\Rightarrow (100)^5 - 5(100)^4 + 10(100)^3 - 10(100)^2 + 500 -$

$\Rightarrow 10000000000 - 500000000 + 10000000$
$100000 + 500 - 1$

$\Rightarrow 9509900499.$

Matrices

Definition:-

A set of (m×n) numbers arranged in the form of an ordered set of 'm rows' and 'n columns' is called m×n matrix.

* $A = [a_{ij}]_{m\times n}$ where a_{ij} represents the element at the intersection of i^{th} row and j^{th} column.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Type of matrices

1. **Square matrix :-** A matrix in which the number of rows is equal to the number of column.

Example :

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 5 & -7 \\ 4 & 0 & 0 \end{bmatrix}_{3\times 3}$$

2. **Horizontal matrix :-** matrix in which number of columns are more than number of rows.

Example : $\begin{bmatrix} 2 & 5 & 7 \\ 1 & -1 & 0 \end{bmatrix}$

3. **Vertical matrix :-** Matrix in which number of rows are more than number of columns.

Example : $\begin{bmatrix} 2 & 1 \\ 5 & -1 \\ 7 & 0 \end{bmatrix}$

4. **Row matrix** :- A matrix having only one row is called a row matrix.

Example : $\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$

5. **Column matrix** :- A matrix having only one column is called a column matrix.

Example $\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

6. **Diagonal matrix** :- A square matrix $A = [a_{ij}]_{n \times n}$ is called a Diagonal matrix if $a_{ij} = 0$ for all $i \neq j$.

A diagonal matrix of order $n \times n$ having $d_1, d_2, \dots d_n$ as diagonal elements is denoted by $diag[d_1, d_2, \dots d_n]$

Example : $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

7. **Scalar matrix** :- A diagonal matrix in which all the diagonal elements are equal.

Example : $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

8. Identity or Unit matrix :- A diagonal matrix in which all diagonal elements are equal to one.

The Identity matrix of order n is denoted by I_n

Example: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$

9. Null matrix :- A matrix whose all elements are zero is called a null matrix or a zero matrix.

Example: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

10. Upper Triangular matrix : A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if $a_{ij} = 0$ for all $i > j$

Example: $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix}$

11. lower triangular matrix :- A square matrix $A = [a_{ij}]$ is called an upper triangular matrix if $a_{ij} = 0$ for all $i < j$

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

Operations of matrices.

1. Equality of matrices :- Two matrices of same order are equal if their corresponding elements are equal.

Example: $\begin{bmatrix} 2x+4 & 3y-x \\ x+2 & x+z \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 3 & -1 \end{bmatrix}$

Find the value of $x+y+z$.

A) 0 B) 1 C) -2 D) -6

2. Addition of matrices :

Addition of two matrices is possible only if they are of same order.

The resultant matrix is obtained by addition of corresponding elements of two given matrices

$A=[a_{ij}]$. $B=[b_{ij}]$
if $C = A+B$
$\Rightarrow [c_{ij}] = [a_{ij}] + [b_{ij}]$

Example !
if $A = \begin{bmatrix} 2 & -1 \\ 0 & 6 \end{bmatrix}_{2\times2}$ $B = \begin{bmatrix} 4 & 7 \\ 2 & -2 \end{bmatrix}_{2\times2}$ Then : A+B

~~A+B~~ Properties of matrix addition

1. Matrix addition is commutative $A+B = B+A$
2. Matrix addition is associative $A+(B+C) = (A+B)+C$
3. Cancellation law holds i.e, if: $A+B = A+C$

3. Substraction of matrices.

Substraction of two matrices is possible only if they are of same order.

The resultant matrix is obtained by substraction of corresponding elements of two given matrices.

$$\boxed{\begin{array}{l} A = [a_{ij}] \quad , B = [b_{ij}] \\ \text{if} \quad C = A - B \\ \Rightarrow [c_{ij}] = [a_{ij}] - [b_{ij}] \end{array}}$$

4. Multiplication of matrix with a scalar.

The scalar gets multiplied in every element of the matrix.

Example : $A = \begin{bmatrix} 2 & -5 & 1 \\ 0 & 6 & -1 \end{bmatrix}_{2\times 3}$

then,

$$2A = \begin{bmatrix} 4 & -10 & 2 \\ 0 & -12 & -2 \end{bmatrix}$$

5. Multiplication of two matrices

Matrix multiplication is done Row by column.

for the two matrices $A_{m\times n}$ and $B_{p\times q}$. the multiplication $(A\times B)$ is possible only if number of columns of A is equal to the number of rows of B. i.e., $n = p$

The resultant matrix C will be of the order $m\times q$

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \\ 5 & 6 \end{bmatrix}_{3\times2}$ and $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 2 \end{bmatrix}_{2\times3}$

Find AB and BA.

Properties of matrix multiplication.

1. In general, $AB \neq BA$. i.e commutative property does not hold. They may be equal for special set of matrices A and B.

2. Multiplication of matrices is distributive w.r.t a addition of matrices $A(B+C) = AB + AC$.

3. Matrix multiplication is associative if conformability is assured $A(BC) = (AB)C$.

4. if $AB = 0$ then it does not neccessarily mean that $A=0$ or $B=0$ or both are 0.

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5. $AI_n = I_nA = A$

Example:

if $A = \begin{bmatrix} a & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

then the value of a for which $A^2 = B$ is

A. 1 B. -1 C. 4 D. No real value

Example: Find the value of x, if $[1\ x\ 1]\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 3 & 2 & 5\end{bmatrix}\begin{bmatrix}+1\\ -2\\ 3\end{bmatrix} = 0.$

The transpose of a matrix.

For the matrix A, the matrix obtained by interchanging rows and columns is called Transpose of matrix A and is denoted by A' or A^T

properties of transpose.

1. $(A')' = A$
2. $K(A') = (KA')$; K being a scalar.
3. $(A+B)' = A' + B'$
4. $(AB)' = B'A'$
5. $(ABC)' = C'B'A'$

Ex: if matrix A, $A = \begin{bmatrix}a & b & c\\ b & c & a\\ c & a & b\end{bmatrix}$ where a, b, c are real positive numbers, $abc = 1$ and $A^TA = I$, then find the value of $a^3+b^3+c^3$.

A) 6 B) 2 C) 4 D) 0

Ex: if P is a 3x3 matrix such that $P^T = 2P + I$ where P^T is the transpose of P and I is the 3x3 identity matrix, then there exists a column matrix $X = \begin{bmatrix}x\\ y\\ z\end{bmatrix} \neq \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$ such that

A) $PX = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$ B). $PX = X$ C) $PX = 2X$ D) P

* Trace of matrix *

if $A = [a_{ij}]_{m \times m}$ be a square matrix then the sum of its diagonal elements is defined as the trace of the matrix A. and is written as $Tr(A)$

properties of trace.

1. $Tr(pA) = p\,Tr(A)$
2. $Tr(pA + qB) = p\,Tr(A) + q\,Tr(B)$.
3. $Tr(AB) = Tr(BA)$, provided AB and BA

Some Special matrices.

1. Symmetric matrix.

Matrix in which $[a_{ij}] = [a_{ji}]$ is called symmetric matrix.

Here, $A' = A$.

2. Skew symmetric matrix:
matrix in which $[a_{ij}] = -[a_{ji}]$ is called skew-symmetric matrix.

Here, $A' = -A$.

properties of symmetric and skew-symmetric matrix

1. In skew symmetric matrix all the diagonal elements are zero.

2. For any matrix $(A+A')$ is always symmetric and $(A-A')$ is skew symmetric.

$$B = A + A^T$$
$$B^T = (A+A^T)^T$$
$$B^T = A^T + (A^T)^T$$
$$B^T = A^T + A.$$

$$C = A - A^T$$
$$C^T = (A - A^T)^T$$
$$C^T = A^T - A$$
$$C^T = -(A - A^T)$$

3. Any matrix can be written as sum of symmetric and skew symmetric matrix.

$$A = \frac{(A+A^T)}{2} + \frac{(A-A^T)}{2}$$

Ex: let X and Y be two arbitrary, 3×3, non-zero, skew symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew-symmetric.

A. $Y^3Z^4 - Z^4Y^3$. C. $X^4Z^3 - Z^3X^4$

B. $X^{44} + Y^{44}$ D. $X^{23} + Y^{23}$

Orthogonal matrix :- A square matrix 'A' is said to be orthogonal if $A^TA = AA^T = I$

$$\boxed{A^TA = AA^T = I}$$

Idempotent matrix : $A^2 = A$

$$\boxed{A^2 = A}$$

Involutory matrix : A matrix 'A' will be called an involutory matrix if $A^2 = I$ (Unit matrix)

$$\boxed{A^2 = I}$$

periodic matrix : A matrix 'A' will be called an periodic matrix if $A^{K+1} = A$ where K is least positive integer for which $A^{K+1} = A$; then K is said to be the period of A.

$$\boxed{A^{K+1} = A \text{ where } K \text{ is period of } A}$$

Nilpotent matrix : A matrix 'A' will be called nilpotent matrix if $A^K = 0$ (null matrix), K is least positive integer and K is called index of the nilpotent matrix.

$$\boxed{A^K = 0,\ K \text{ is index of } A}$$

Ex :- if A, B are two Idempotent matrices and $AB = BA = 0$ then $A+B$ is

A. Involutory matrix B. Idempotent matrix

C. orthogonal matrix D. Nilpotent matrix

Ans :- $A^2 = A$; $B^2 = B$

$$(A+B)^2 = (A+B)(A+B)$$

$$= A^2 + AB + BA + B^2$$

$$= A^2 + B^2$$

Ex:- Let A is a matrix of order 2x2 such that $A^2=0$. Then Tr(A) is:

A) 1 B) 0 C) -1. D) None

Ans:
$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}; \quad A^2=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^2=\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= b(a+d)=0 \qquad a+d\neq 0 \Rightarrow b=0 \; \& \; c=0.$$
$$c(a+d)=0 \qquad \because a^2+bc=0$$
$$\Rightarrow a=0, \; bc+d^2=0 \Rightarrow d=0$$

—: DETERMINANTS :—

Determinant:

It is a number.

It is defined only for a square matrix.

Determinant of matrix A is written as $|A|$

Minors:-

minors of an element $[a_{ij}]$ is expressed as M_{ij} and is obtained by removing i^{th} row and j^{th} column

Ex:- Find minor M_{23} in the following terms:

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & -1 & 2 \end{vmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$$

$$-1-8=-9.$$

Cofactor:

Cofactor of an element $[a_{ij}]$ is expressed as C_{ij} and is calculated as:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Ex:- Find cofactor C_{23}: $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & -1 & 2 \end{vmatrix}$

$$M_{23} = \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix}$$

$$M_{23} = -1-8 = -9.$$

$$C_{23} = (-1)^{2+3} M_{23}$$

$$C_{23} = -1 \times -9 = 9$$

Calculation of Determinant

It is the sum of product of elements of a row (or column) with their corresponding cofactors.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Ex:- Evaluate the following determinants:

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & -1 & 2 \end{vmatrix}$$

Ex:- if $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$; then

A. $x = 3, y = 1$ B. $x = 1, y = 3$

C. $x = 0, y = 3$ D. $x = 0, y = 0$

Multiplication of scalar to a determinant

The scalar (number) gets multiplied in any one row or column.

Ex:- let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$ where $[b_{ij}] = 2^{i+j}[a_{ij}]$ for $1 < i, j < 3$. if the determinants of P is 2. then the determinants of the matrix of Q is.

A 2^{10} B. 2^{11} C. 2^{12} D. 2^{13}.

$$|P| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \Rightarrow |Q| = 2^{i+j}|P|$$

$$= \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix}$$

$|Q| = (2^2)(2^3)(2^4)(2)(2^2)|P|$

$|Q| = 2^{12} \times |P|$

$= 2^{12} \times 2$

$|Q| = 2^{13}$

properties of Determinants:-

1. Determinant is zero if any row or column is zero.
2. Determinant is zero if two rows or column have equal or proportional values.
3. Value of determinant changes by a minus sign if we exchange two rows or two columns.
4. Elementary row or column transformation do not change the value of determinant

$R_1 \to R_1 + R_3 \to R_3 + mR_2$.

5. $|AB| = |A| |B|$
6. $|A^T| = |A|$
7. $|KA| = K^n |A|$; n is order of A.

Important results.

1. Determinant of skew symmetric matrix of odd order is zero.
2. Determinant of diagonal matrix is equal to the product of its diagonal elements.
3. If a number is multiplied to a determinant, it gets multiplied in any one row or column.

Singular and non-singular matrix: matrix whose determinant is zero is called singular matrix otherwise it is Non-singular matrix.

Ex:- $\begin{vmatrix} 2c & 2c & c-a-b \\ a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \end{vmatrix}$

A. $(a+b+c)^2$ B) $(a+b+c)^3$ C) $(a+b+c)$ D) $(a+b+c)^4$

Ex:- The determinant $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}$ is equal to zero if

A. a, b, c are in AP
B. a, b, c are in GP
C. a, b, c are in HP
D. α is a root of the equation $ax^2+2bx+c$.

Ex: if a, b, c are sides of a triangle and

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 0, \text{ then}$$

A. ΔABC is equilateral
B. ΔABC is a Isosceles
C. ΔABC is a right angled triangle
D. ΔABC is a Scalene triangle.

—: Inverse of a Matrix :—

let A be any n-rowed square matrix. Then a matrix B, if exists such that $AB = BA = I_n$ is called the inverse of A.

inverse of A is usually denoted by A^{-1} (if exists)

$$A(\text{adj } A) = |A| I$$

$$A\left(\frac{1}{|A|}\text{adj } A\right) = I$$

properties of Inverse matrix

1. Every invertible matrix possess a unique inverse
2. if A and B are invertible matrices of the same order, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
3. if A is invertible square matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$
4. $|A^{-1}| = |A|^{-1}$

Ex: if $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1}

Ex: $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & a \\ 1 & -2 & 3 \end{bmatrix}$

if B is inverse of A then a is :-

Inverse using elementary transformation

Ex:- if $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1}

Cayley - Hamilton Theorem.

Every square matrix satisfies a specific polynomial equation known as characteristic equation.

$$P(\lambda) = |A - \lambda I|$$
$$P(A) = 0$$

$$a_1 \lambda^3 + b_1 \lambda^2 + c_1 \lambda + d_1 = 0$$
$$a_1 A^3 + b_1 A^2 + c_1 A + d_1 I = 0$$

Ex! if $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, prove that $A^2 - 4A - 5I = 0$

Ex:- $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$$A^{-1} = \left[\frac{1}{6}(A^2 + cA + dI)\right]$$

Then the value of c and d are:

A (-6, -11) . B. (6, 11) . C (-6, 11) D (6, -1

System of linear Simultaneous Equations:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

I. $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ Unique Solution

II. $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ No solution

III. $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ Infinite solution.

Ex: The number of values of K for which the system of equations: $(K+1)x + 8y = 4K$; $Kx + (K+3)y = 3K-1$ has Infinitely many solution is.

A 0 B 1 C 2 D Infinite

Ans: $(K+1)x + 8y = 4K$
$Kx + (K+3)y = 3K-1$

$$\frac{K+1}{K} = \frac{8}{K+3} = \frac{4K}{3K-1}$$

to get ans: $K = 1$ & $K = 3$.

System of linear simultaneous Equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\Rightarrow AX = B$$

Method 1   Matrix method

a) $\Delta \neq 0$, then A^{-1} exists

$$\Rightarrow A^{-1}(AX) = A^{-1}B$$
$$\Rightarrow X = A^{-1}B$$

and therefore unique values of x, y and z are obtained.

b) $\Delta = 0$ then A^{-1} does not exists

We have $AX = B$

$$\Rightarrow ((\text{adj } A)A)X = (\text{adj } A)B$$
$$\Rightarrow \Delta x = (\text{adj } A)B.$$

1. $(\text{adj } A)B = 0$, then the system $AX = B$ has infinitely many solution.

2. $(\text{adj } A)B \neq 0$ then the system $AX = B$ has no solution.

Determinant method of solution.

$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
$a_3x + b_3y + c_3z = d_3$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\boxed{AX = B}$$

Method 2 (Cramer's Rule)

let $|A| = \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

obtained on replacing first column of Δ by B

Similiarly, $\Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $\Delta z = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$x \cdot \Delta = \Delta x \qquad y \cdot \Delta = \Delta y \qquad z \cdot \Delta = \Delta z$

Case 1

if $\Delta \neq 0$, then the given systems of equations has unique solutions given by

$$x = \frac{\Delta x}{\Delta}, \quad y = \frac{\Delta y}{\Delta} \text{ and } z = \frac{\Delta z}{\Delta}$$

Case 2

if $\Delta = 0$, then two sub cases arise :

a) At least one of Δx, Δy and Δz is non-zero

$$x \cdot \Delta = \Delta x \quad y \cdot \Delta = \Delta y, \quad z \cdot \Delta = \Delta z$$

(No solution)

Case 3

if $\Delta = 0$, then two sub cases arise

b) All of Δx, Δy and Δz are zero

Either No solution or Infinite solution

Ex:- Given

$$2x - y + 2z = 2$$
$$x - 2y - z = -4$$
$$x + y + \lambda z = 4$$

then the value of λ, such that the given system of equation has No solution

A. 3 B. 1 C. 0 D. -1

Solution of homogeneous simultaneous equations

$a_1x + b_1y + c_1z = 0$
$a_2x + b_2y + c_2z = 0$
$a_3x + b_3y + c_3z = 0$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Ex: The system of equation:

$\lambda x + y + z = 0$
$-x + \lambda y + z = 0$
$-x - y + \lambda z = 0$

will have a non-zero solution if real values of λ are given by:

$$\Delta = \begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$$

$\lambda(\lambda^2+1) - 1(-\lambda+1) + 1(1+\lambda) =$
$\lambda^3 + \lambda + \lambda - 1 + 1 + \lambda = 0$
$\lambda^3 + 3\lambda = 0$