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BASIC MATHEMATICS (SEM. - I)

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Chapter 1

LOGARITHMS

1.1 INTRODUCTION

A power function has the form

$$y = a^x \quad \dots (1)$$

We often say that 'a' is raised to the power x. 'a' is known as the **base,** while 'x' is called the **power** or **exponent** and the power function means 'a' is multiplied by itself x-times.

Thus,

$$y = a^x$$
$$= a \times a \times \dots x \text{ times}$$

For example,

$$y = 2^3$$
$$= 2 \times 2 \times 2 \text{ (3 times)}$$
$$= 8$$
$$\therefore \quad 2^3 = 8$$

Inversely, if we are given the base 2 and its power 8, that is,

$$2^? = 8$$

then what is the exponent that will produce 8? We know from the laws of indices that 3 is the exponent to which 2 must be raised to produce 8. But in case $2^x = 5.5$, then we cannot answer the value of x by merely tables or inspection.

Here other exponent x is called logarithm. We call the exponent x the logarithm of 5.5 with base 2.

We write

$$x = \log_2 5.5$$

We write the base 2 as a subscript.

Definition :

Thus, *the inverse of the power function is called a logarithm OR a logarithm is an exponent.*

Given $y = f(x)$, we often need to find $x = g(y)$ that is in present case, from equation (1)

$$x = \log_a y \qquad \ldots (2)$$

y is called the **argument** of the logarithm. $x = \log_a y$ if and only if $a^x = y$.

These alternative forms are shown in **Fig. 1.1** and are equivalent. They are inter-convertible. No... positions of the different quantities in these two alternative forms.

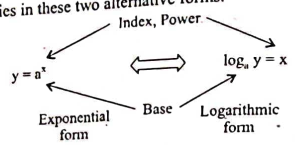


Fig. 1.1

Thus, for example, if

(i) $10^2 = 100$ then $\log_{10} 100 = 2$

(ii) $10^{-1} = \frac{1}{10}$ then $\log_{10} \frac{1}{10} = -1$

(iii) $\log_5 \frac{1}{25} = -2$ then $5^{-2} = \frac{1}{25}$.

Remember that exponential and logarithmic functions are one-to-one functions. That is they ha... inverses. We can always write

$$y = a^{\log_a y} \qquad \ldots (3)$$

which emphasizes that a logarithm is just an exponent of a power function.

The following rules of exponent are extremely useful in understanding logarithm in-depth. For an... numbers a, x and y, we have :

(1) $a^x \cdot a^y = a^{x+y}$

(2) $\frac{a^x}{a^y} = a^{x-y}$ if $x > y$

$= \frac{1}{a^{y-x}}$ if $y > x$

(3) $(a^x)^y = (a^y)^x = a^{xy}$

(4) $a^{-x} = \frac{1}{a^x}$

(5) $a^{x/y} = (a^{1/y})^x$

(6) $a^0 = 1$

(7) $a^1 = a$

1.2 TWO BASIC PROPERTIES OF LOGARITHM

1.2.1 To Show That $\log_a 1 = 0$

We have $a^0 = 1$

$\therefore \quad \log_a 1 = 0$

Any positive number except 1 may be selected as the base. Thus,

the logarithm to any base of 1 is zero.

For example : $\log_{10} 1 = 0, \ \log_3 1 = 0, \ \log_{2\sqrt{2}} 1 = 0$

1.2.2 To Show That $\log_a a = 1$

We have $a^1 = a$

$\therefore \quad \log_a a = 1$

Thus,

The logarithm of the base itself is 1.

1.3 COMMON LOGARITHMS

In science, large and small numbers are often expressed in scientific notation. For example,

$1000 = 10^3$

$3510 = 3.510 \times 10^3$

For this reason powers of 10 are very important and we commonly want to calculate the exponent on such a power. For example,

$x = \log_{10} 10^3 = 3$

and $\quad x = \log_{10} 3510 = 3.5453$

The system of common logarithms has 10 as its base. In other words, the base 10 logarithm is called common logarithm.

It is written as $\log_{10} a$.

Often the subscript, that is, base is typically dropped. Thus,

log a means $\log_{10}$ a.

Similarly $\log 100 = 2$ implies that the base is 10.

Here are the powers of 10 and their logarithms.

Powers of 10 :	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100	1000	10000
Logarithms :	−3	−2	−1	0	1	2	3	4

A scientific calculator may be required to evaluate such expressions. Note that $\log_{10}$ key is usually implemented as **inv 10^x** or vice-versa.

1.4 NATURAL LOGARITHMS

The system of natural logarithms has the number called "e" as its base. "e" is named after the 18th century swiss mathematician Leonhard Euler. "e" is the base used in calculus. Since the number "e" is arises from the description of naturally occurring processes (radioactive decay), the logarithmic function based on powers of "e" is often known as the *natural logarithm.* "e" can be calculated from the following series involving factorials.

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \ldots$$

Note that $n! = 1.2.3. \dots (n-1) \cdot n$

$\therefore$ $3! = 1.2.3 = 6$, $4! = 1.2.3.4 = 24 \dots$ so on.

"e" is an irrational number whose decimal value is approximately.

2.71828182845904

The base 'e' logarithm which is the inverse of e^x is called the natural logarithm.

To write it, we use the notation

"ln" or "$\log_e$"

ln x means **$\log_e x$**

This function can also be found on a sciеitific calculator in the same way as finding $\log_{10} x$. The key is usually marked $\log_e x$ or ln x. You may have to press 'Inv' followed by e^x if your calculator doesn't have log e for ln key.

The exponential function can be used to define any other function, power function, because any base can be expressed as a power of e with a scaled exponent

$a^x = e^{x \ln a}$.

1.5 RELATION BETWEEN COMMON AND NATURAL LOGARITHMS

Common and natural logarithms can be expressed in terms of each other as

$$\log_e N = \frac{\log_{10} N}{\log_{10} e} \quad \dots \text{By the rule of change of base}$$

Since $\log_{10} e = 0.4343$ and $\frac{1}{\log_{10} e} = \frac{1}{0.4343} = 2.303$

$\therefore$ **$\log_e N = 2.303 \log_{10} N$** where N is a positive number.

The natural logarithm is especially useful in calculus because its derivative is given by the simple equation

$$\frac{d}{dx}[\log_e x] = \frac{1}{x}.$$

1.6 LAWS OF LOGARITHMS

If $a > 0$, $a \neq 1$, 'm' and 'n' are positive real numbers, then we have the following rules of logarithms. The laws of logarithms will be valid for any base, we will prove them for base a.

1.6.1 Logarithm of Product

We have

$$\log_a (m \times n) = \log_a m + \log_a n$$

"The logarithm of a product is equal to the sum of the logarithms of each factor".

Proof :

The function $y = \log_a x$ is defined for all positive real numbers x. Therefore there are real numbers p and q such that

$p = \log_a m$ and $q = \log_a n$

This implies

$m = a^p$ and $n = a^q$

Therefore, according to the laws of exponents,

$$m \times n = a^p \times a^q = a^{p+q}$$

$$\therefore \quad \log_a (m \times n) = \log_a a^{p+q} = p + q = \log_a m + \log_a n$$

Thus,

$$\log_a (m \times n) = \log_a m + \log_a n$$

Corollary : It follows that

$$\log_a (m \times n \times r \times s \times \dots) = \log_a m + \log_a n + \log_a r + \log_a s + \dots$$

The converse is useful in expressing the statement in a single logarithm.

1.6.2 Logarithm of Quotient

We have

$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$$

"The logarithm of a quotient is equal to the logarithm of numerator minus the logarithm of the denominator."

Let $\log_a m = p$ and $\log_a n = q$

$\therefore$ $m = a^p$ and $n = a^q$

$$\frac{m}{n} = \frac{a^p}{a^q} = a^{p-q} \quad \dots \text{By law of indices}$$

$$\therefore \quad \log_a \left(\frac{m}{n}\right) = \log_a (a^{p-q}) = p - q$$

Thus,

$$\therefore \quad \log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n.$$

Corollary I :

$$\log_a \left(\frac{m \times n}{r \times s}\right) = \log_a (m \times n) - \log_a (r \times s) = \log_a m + \log_a n - \log_a r - \log_a s$$

Conversely, *while writing sum or difference of logarithms into a single logarithm, the positive terms are multiplied in numerator and the negative terms are multiplied in denominator.*

For example :

$$\log 2 - \log 3 + \log 8 - \log 4 = \log \left\{\frac{2 \times 8}{3 \times 4}\right\} = \log \left(\frac{4}{3}\right)$$

Corollary II :

$$\log_a\left(\frac{1}{n}\right) = \log_a 1 - \log_a n = 0 - \log_a n = -\log_a n$$

Thus,

$$\log_a\left(\frac{1}{n}\right) = -\log n$$

It follows that $\log_a\left(\frac{m}{n}\right) = -\log_a\left(\frac{n}{m}\right)$

"The logarithm of inversion of a fraction changes by sign only."

For example, $\log_2\left(\frac{1}{8}\right) = -\log_2 8$

1.6.3 Logarithm of Power

We have

$$\log_a m^n = n \cdot \log_a m$$

"The logarithm of a power of m is equal to the exponent of that power times the logarithm of m."

Proof :

There is a real number p such that

$$p = \log_a m$$

$$\therefore \quad m = a^p$$

The rules of exponents are valid for all rational numbers n.

$\therefore \quad m^n = (a^p)^n = a^{pn}$

This implies $\log_a m^n = \log_a a^{pn}$

$= pn$

$= np$

$= n \cdot \log_a m$

$\therefore \quad \log_a m^n = n \cdot \log_a m$

For example,

$\log_4 16 = \log_4 4^2$ ... $\because 16 = 4^2$

$= 2 \cdot \log_4 4$ ... By this rule

$= 2(1)$ ... $\log_a a = 1$

$= 2$

1.6.4 Rule of Change of Base

We know the values of logarithms of base 10, but not for any other base. In this case, we convert a logarithm in base 10 to one in base, say 2, or any other base by realizing that the values will be proportional.

Thus,

$$\log_2 x = k \cdot \log_{10} x \quad \ldots (1)$$

Each value in base 2 will differ from the value in base 10 by the same constant k.

Now, to find that constant, we know that

$$\log_2 2 = 1$$

Therefore, on putting $x = 2$ in equ. (1) above, we get

$\therefore \quad \log_2 2 = k \cdot \log_{10} 2$

$\therefore \quad 1 = k \cdot \log_{10} 2$

$\therefore \quad k = \frac{1}{\log_{10} 2}$

Substituting this k in equ. (1), we get

$$\log_2 x = \frac{\log_{10} x}{\log_{10} 2}$$

Now, by knowing the values in base 10, we can in this way find the values in base 2.

In general, if we know the values in base a, then we can change to base b as follows :

$$\log_b x = \frac{\log_a x}{\log_a b}, \ x \neq 1, b \neq 1, x \text{ and } b \text{ are positive real numbers.}$$

This statement is known as the rule of change of base.

Corollary I : Putting $x = a$ in the above expression, we get

$\therefore \quad \log_b a = \frac{\log_a a}{\log_a b}$

But $\log_a a = 1$

$\therefore \quad \log_b a = \frac{1}{\log_a b}$ OR $\log_b a \times \log_a b = 1$

For instance,

(i) $\log_e 10 = \frac{1}{\log_{10} e}$

(ii) $\frac{1}{\log_8 4} = \log_4 8$.

Corollary II :

$$\log_{a^n} x^n = \log_a x$$

For instance $\log_{27} 8 = \log_{3^3} 2^3 = \log_3 2$

We will practice the use of all these laws in the following illustrative examples :

The following facts will be helpful solving problems on logarithms :

(1) $\log_a a^x = x$

(2) $\log_{10} 10^x = x$

(3) $\log_e e^x = x$

(4) $a^{\log_a x} = x$

(5) $10^{\log_{10} x} = x$

(6) $e^{\log_e x} = x$

(7) $\log a = \log b$ if and only if $a = b$. The converse is also true.

(8) The following table of powers is a tool for quick answering. Therefore, students are advise to remember them through practice rather than merely memorizing.

Index of 2	Index of 3	Index of 4	Index of 5	Index of 6	Index of 7
$2^0 = 1$	$3^0 = 1$	$4^0 = 1$	$5^0 = 1$	$6^0 = 1$	$7^0 = 1$
$2^1 = 2$	$3^1 = 3$	$4^1 = 4$	$5^1 = 5$	$6^1 = 6$	$7^1 = 7$
$2^2 = 4$	$3^2 = 9$	$4^2 = 16$	$5^2 = 25$	$6^2 = 36$	$7^2 = 49$
$2^3 = 8$	$3^3 = 27$	$4^3 = 64$	$5^3 = 125$	$6^3 = 216$	$7^3 = 343$
$2^4 = 16$	$3^4 = 81$	$4^4 = 256$	$5^4 = 625$		
$2^5 = 32$	$3^5 = 243$		$5^5 = 3125$		
$2^6 = 64$	$3^6 = 729$				
$2^7 = 128$					
$2^8 = 256$					
$2^9 = 512$					

(9) **Common Mistakes : Be Careful !!**

(i) $\log_a (x \pm y) \neq \log_a x \pm \log_a y$

(ii) $(\log_a x)^n \neq n \cdot \log_a x$

(iii) $\log_a \left(\frac{x}{y}\right) \neq \frac{\log_a x}{\log_a y}$

.... ILLUSTRATIVE EXAMPLES ....

(1) Write the following results into their equivalent logarithmic form :

(a) $2^5 = 32$ (b) $7^0 = 1$

(c) $10^{-2} = \frac{1}{100}$ (d) $9^{-1/2} = \frac{1}{3}$.

Solution :

(a) $2^5 = 32 \Rightarrow \log_2 32 = 5$

(b) $7^0 = 1 \Rightarrow \log_7 1 = 0$

(c) $10^{-2} = \frac{1}{100} \Rightarrow \log_{10} \frac{1}{100} = -2$

(d) $9^{-1/2} = \frac{1}{3} \Rightarrow \log_9 \frac{1}{3} = -\frac{1}{2}$

(2) Write the following results into their equivalent exponential form :

(a) $\log_7 343 = 3$ (b) $\log_3 3 = 1$

(c) $\log_6 \frac{1}{36} = -2$ (d) $\log_{81} \frac{1}{3} = -\frac{1}{4}$.

Solution :

(a) $\log_7 343 = 3 \Rightarrow 7^3 = 343$

(b) $\log_3 3 = 1 \Rightarrow 3^1 = 3$

(c) $\log_6 \frac{1}{36} = -2 \Rightarrow 6^{-2} = \frac{1}{36}$

(d) $\log_{81} \frac{1}{3} = -\frac{1}{4} \Rightarrow (81)^{-1/4} = \frac{1}{3}$

(3) Evaluate or find the values of the following :

(a) $\log_5 125$ (b) $\log_{64} 4$

(c) $\log_{16} 32$ (d) $\log_{2\sqrt{3}} 12$

(e) $\log_9 \frac{1}{81}$ (f) $\log_5 (\log_3 3)$

(g) $\log_b \sqrt{b} - \log_a \frac{1}{a}$.

Solution :

(a) Let $\log_5 125 = x$ (Logarithmic form)

$\therefore$ $5^x = 125$ (Exponential form)

$\therefore$ $5^x = 5^3$ ... From indices

$\therefore$ $x = 3$

$\therefore$ $\log_5 125 = 3$

Note this problem can also be solved as below. But the first method is more easy.

$\log_5 125 = \log_5 5^3$

$= 3 \cdot \log_5 5$ ... By $\log_a a^n = n \log_a a$

$= 3 (1)$ ... By $\log_a a = 1$

$= 3$

Both above methods are valid. Use that which you find more convenient.

(b) Let $\log_{64} 4 = x$ (Logarithmic form)

$\therefore$ $64^x = 4$ (Exponential form)

$\therefore$ $(4^3)^x = 4$

$\therefore$ $4^{3x} = 4^1$ ... By indices

$\therefore$ $3x = 1$

$\therefore$ $x = \frac{1}{3}$

$\therefore$ $\log_{64} 4 = \frac{1}{3}$

(c) Let $\log_{16} 32 = x$ (Logarithmic form)

$\therefore$ $16^x = 32$ (Exponential form)

$\therefore$ $(2^4)^x = 2^5$

$\therefore$ $2^{4x} = 2^5$ ... By $(a^m)^n = a^{mn}$

$\therefore$ $4x = 5$

$\therefore$ $x = \frac{5}{4}$

$\therefore$ $\log_{16} 32 = \frac{5}{4}$

(d) Let $\log_{2\sqrt{3}} 12 = x$ (Logarithmic form)

$\therefore$ $(2\sqrt{3})^x = 12$ (Exponential form)

Observe that $(2\sqrt{3})^2 = 4 \times 3 = 12$

$(2\sqrt{3})^x = (2\sqrt{3})^2$

$\therefore \quad x = 2$

$\therefore \quad \log_{2\sqrt{3}} 12 = 2$

(e) Let $\log_9 \frac{1}{81} = x$ (Logarithmic form)

$9^x = \frac{1}{81}$ (Exponential form)

$\therefore \quad (3^2)^x = (3^{-4})^1$ ... Converting on both sides into common base

$\therefore \quad 3^{2x} = 3^{-4}$ ... By $(a^m)^n = a^{mn}$

$\therefore \quad 2x = -4$

$\therefore \quad x = -2$

$\therefore \quad \log_9 \frac{1}{81} = -2$

(f) $\log_5 (\log_3 3)$

Observe that $\log_3 3 = 1$

$\therefore \quad \log_5 (\log_3 3) = \log_5 1$

$= 0$ ... By $\log_a 1 = 0$

$\therefore \quad \log_5 (\log_3 3) = 0$

(g) $\log_b \sqrt{b} - \log_a \frac{1}{a} = \log_b b^{1/2} + \log_a a$ ... By $\log\left(\frac{1}{n}\right) = -\log n$

$= \frac{1}{2} \log_b b + 1$

$= \frac{1}{2} + 1$ ... By $\log_a a = 1$

$= \frac{3}{2}$

(4) Express the following as a single logarithm in their simplest forms :

(a) $\log 5 + \log 3 - \log 2$ **(b) $\frac{1}{2} \log 9 + \frac{1}{3} \log 27$**

(c) $2 \log_{10} 4 - \frac{1}{2} \log_{10} 16 + 1$ **(d) $\log (x^2 - 3x + 2) - \log (x - 1) + \log (x - 2)$.**

Solution :

(a) $\log 5 + \log 3 - \log 2 = \log\left(\frac{5 \times 3}{2}\right) = \log \frac{15}{2}$

(b) $\frac{1}{2} \log 9 + \frac{1}{3} \log 27$

$= \log 9^{1/2} + \log 27^{1/3}$ ... By $n \log a = \log a^n$

$= \log 3 + \log 3$

$= \log (3 \times 3)$

$= \log 9$

(c) $2 \log_{10} 4 - \frac{1}{2} \log_{10} 16 + 1$

$= \log_{10} 4^2 - \log_{10} 16^{1/2} + \log_{10} 10$ ... Note that $1 = \log_{10} 10$ as base 10 is in other terms.

$= \log_{10} 16 - \log_{10} 4 + \log_{10} 10$

$= \log_{10} \left(\frac{16 \times 10}{4}\right)$

$= \log_{10} 40$

(d) $\log (x^2 - 3x + 2) - \log (x - 1) + \log (x - 2)$

$= \log\left[\frac{(x^2 - 3x + 2)(x - 2)}{x - 1}\right]$

$= \log\left[\frac{(x - 2)(x - 1)(x - 2)}{x - 1}\right]$

$= \log (x - 2)^2$

$= 2 \cdot \log (x - 2)$

(5) Solve for x :

(a) $\log_5 x = 2$ **(b) $\log_2 \frac{1}{2} = x$**

(c) $\log_4 (x - 2) = 2$ **(d) $\log_5 (x^2 - 5x + 11) = 1$**

(e) $\log_2 x - \log_2 (x - 1) = 5$ **(f) $\log x + \log (x + 1) = \log 6$.**

Solution :

(a) Given $\log_5 x = 2$

Converting it into exponential form, we get

$\therefore \quad x = 5^2 = 25$

(b) Given $\log_2 \frac{1}{2} = x$

Converting it into exponential form, we get,

$\therefore \quad 2^x = \frac{1}{2} = 2^{-1}$

$\therefore \quad x = -1$

(c) Given $\log_4 (x - 2) = 2$

Converting it into exponential form, we get

$\therefore \quad x - 2 = 4^2$

$\therefore \quad x - 2 = 16$

$\therefore \quad x = 18$

(d) Given $\log_5 (x^2 - 5x + 11) = 1$

Converting it into exponential form, we get

$\therefore \quad x^2 - 5x + 11 = 5^1$

$\therefore \quad x^2 - 5x + 6 = 0$ (Quadratic in x)

Factorising, $(x-2)(x-3) = 0$

$\therefore \quad x = 2$ or $x = 3$

Check : Substituting $x = 2$ in original equation

$\log_5 (4 - 10 + 11) = \log_5 5 = 1$, satisfied

Substituting $x = 3$,

$\log_5 (9 - 15 + 11) = \log_5 5 = 1$, satisfied.

$\therefore$ Both $x = 2$ and $x = 3$ are the solution of the equation.

(e) Given $\log_2 x - \log_2 (x-1) = 5$

$$\therefore \quad \log_2 \left(\frac{x}{x-1}\right) = 5$$

Converting it into exponential form, we get

$$\therefore \quad \frac{x}{x-1} = 2^5$$

$$\therefore \quad x = 32(x-1)$$

$$\therefore \quad x = 32x - 32$$

$$\therefore \quad -31x = -32$$

$$\therefore \quad x = \frac{32}{31}.$$

(f) Given $\log x + \log (x+1) = \log 6$

$\therefore \quad \log x(x+1) = \log 6$

Dropping logarithm from both sides having same base,

$\therefore \quad x(x+1) = 6$

$\therefore \quad x^2 + x - 6 = 0$

Factorising we get

$\therefore \quad (x+3)(x-2) = 0$

$\therefore \quad x = -3$ or $x = 2$

Check : Putting $x = -3$ in original equation.

$\log(-3) + \log(-2) = \log 6$ is not defined as log (negative number) does not exist.

$\therefore \quad x = -3$ is rejected.

Putting $x = 2$ in original equation.

$\log 2 + \log (3) = \log 6$ is true.

$\therefore \quad x = 2$ is the required solution.

(6) Solve for x :

$$\log(1+x) - \log(1-x) = 1.$$

Solution :

$$\therefore \quad \log \frac{1+x}{1-x} = 1$$

If we let each side be the exponent with base e, then

$$\frac{1+x}{1-x} = e = \frac{e}{1}$$

$$\therefore \quad 1 + x = e - ex$$

$$\therefore \quad x + ex = e - 1$$

$$\therefore \quad (1+e)x = e - 1$$

$$\therefore \quad x = \frac{e-1}{e+1}$$

(7) Solve $2^{2x} - 5(2^{x+1}) + 16 = 0$.

Solution : Given $2^{2x} - 5 \cdot 2^{x+1} + 16 = 0$

Rewriting using laws of indices, we get

$\therefore \quad (2^x)^2 - 5 \cdot 2^x \cdot 2 + 16 = 0$

Let $\quad 2^x = y$

$\therefore \quad y^2 - 10y + 16 = 0$ (Quadratic in y)

Factorizing, $(y-8)(y-2) = 0$

$\therefore \quad y = 8$ or $y = 2$

But $y = 2^x$

$\therefore \quad 2^x = 8$ or $2^x = 2$

$\therefore \quad 2^x = 2^3$ or $2^x = 2^1$

$\therefore \quad x = 3$ or $x = 1$

$\therefore$ Solution is $x = 3$ or $x = 1$.

(8) Evaluate : $\frac{1}{\log_5 10} + \frac{1}{\log_{20} 10}$.

Solution : By rule of change of base, we have

$$\frac{1}{\log_5 10} + \frac{1}{\log_{20} 10} = \log_{10} 5 + \log_{10} 20$$

$= \log_{10} (5 \times 20)$ Converting into single logarithm

$= \log_{10} 100$

$= \log_{10} 10^2$

$= 2 \cdot \log_{10} 10$ ... By $\log_a m^n = n \log_a m$

$= 2(1)$ ... By $\log_a a = 1$

$= 2$

(9) Prove that

$\log_b a \times \log_c b \times \log_a c = 1$, where a, b, and c are all positive numbers.

Solution :

L.H.S. $= \log_b a \times \log_c b \times \log_a c$

By rule of change of base,

$$= \frac{\log a}{\log b} \times \frac{\log b}{\log c} \times \frac{\log c}{\log a}$$

$$= 1$$

$$= \text{R.H.S.}$$

(10) Solve $(x^2 - 3x + 1)^{x^2-2x} = 1$.

Solution : This problem could be solved using the fact of indices.

Using $a^0 = 1$ and $1^x = 1$

$\therefore\ x^2 - 2x = 0$	$x^2 - 3x + 1 = 1$
$\therefore\ x(x-2) = 0$	$\therefore\ x^2 - 3x = 0$
$\therefore\ x = 0$ or $x = 2$	$\therefore\ x = 0$ or $x = 3$

Combining the two results, we have

$x = 0, 2, 3.$

(11) Find the value using rules of logarithm :

$$2\log\left(\frac{3}{4}\right) + \log\left(13\frac{1}{3}\right) - \log\left(7\frac{1}{2}\right).$$

Solution : Given : $2\log\left(\frac{3}{4}\right) + \log\left(13\frac{1}{3}\right) - \log\left(7\frac{1}{2}\right)$

Converting into single logarithm followed by simplification, we get

$$= \log\left(\frac{3}{4}\right)^2 + \log\left(\frac{40}{3}\right) - \log\left(\frac{15}{2}\right)$$

$$= \log\frac{9}{16} + \log\frac{40}{3} - \log\frac{15}{2}$$

$$= \log\left[\frac{\frac{9}{16} \times \frac{40}{3}}{\frac{15}{2}}\right]$$

$$= \log\left[\frac{9 \times 40 \times 2}{16 \times 3 \times 15}\right]$$

$$= \log 1$$

$$= 0$$

(12) Solve : $\log_2 x + \log_4 x = 2$.

Solution : Given $\log_2 x + \log_4 x = 2$

By the rule of change of base, we gt

$$\therefore\ \frac{\log x}{\log 2} + \frac{\log x}{\log 4} = 2$$

$$\therefore\ \frac{\log x}{\log 2} + \frac{\log x}{2 \cdot \log 2} = 2 \qquad \text{... Note that } \log 4 = \log 2^2 = 2 \cdot \log 2$$

$$\therefore\ \frac{\log x}{\log 2}\left[1 + \frac{1}{2}\right] = 2$$

$$\therefore\ \log_2 x = 2 \times \frac{2}{3}$$

$$\therefore\ \log_2 x = \frac{4}{3}$$

$$\therefore\ x = 2^{4/3} = (2^4)^{1/3} = 16^{1/3} = \sqrt[3]{16}.$$

(13) Given $\log_y 3 = p$, $\log_y 5 = q$,

Find : (i) $\log_{15} y$, (ii) $\log_y 15y^2$.

Solution : Given : $\log_y 3 = p$, $\log_y 5 = q$

(i) $\log_{15} y = \dfrac{1}{\log_y 15}$ ... By rule of change of base

$$= \frac{1}{\log_y (3 \times 5)}$$

$$= \frac{1}{\log_y 3 + \log_y 5}$$

$$= \frac{1}{p + q}$$

(ii) $\log_y 15y^2 = \log_y (3 \times 5 \times y^2)$

$$= \log_y 3 + \log_y 5 + 2\log_y y$$

$$= p + q + 2(1)$$

$$= p + q + 2$$

.... EXERCISES ....

(1) Write each of the following in logarithmic form :

(a) $2^3 = 8$ (b) $8^0 = 1$

(c) $10^{-1} = 0.1$ (d) $4^{-2} = \frac{1}{16}$

(e) $27^{-1/3} = \frac{1}{3}$.

(2) Write each of the following in exponential form :

(a) $\log_q p = r$ (b) $\log_2 32 = 5$

(c) $\log_{10} 0.01 = -2$ (d) $\log_{1/2} \frac{1}{8} = 3$

(e) $\log_{27} 9 = \frac{2}{3}$.

(3) Evaluate the following :

(a) $\log_2 8\sqrt{2}$ (b) $\log_{48} 4\sqrt{3}$

(c) $\log_2 \frac{1}{256}$ (d) $\log_{10} 0.001$

(e) $\log_{81} 27$ (f) $\log_8 16$

(g) $\log_b \sqrt[4]{b^3}$

(h) $\log_2 (\log_x x^2)$

(i) $\log_2 \sqrt{\frac{32}{\sqrt{8}}}$.

(4) Convert the following into single logarithm :

(a) $2 \lg o\ 3 + 3 \log 2$

(b) $2 \log 5 + 3 \log 4 - 4 \cdot \log 2$

(c) $\log_{10} 2 + 3$

(d) $\log \frac{1}{2} - \log \frac{5}{6} + \log \frac{3}{4} - \log \frac{7}{8}$.

(5) Solve the following equations for x :

(a) $\log_3 x + \log_3 4 = 2$

(b) $\log_2 (x^2 + 7) = 4$

(c) $\frac{3}{2} \log 4 - \frac{2}{3} \log 8 + \log 2 = \log x$

(d) $\log_2 x + \log_2 (x + 2) = 3$

(e) $\log_{10} (x^2 + 6x + 28) = 2$

(f) $\log (x + 3) + \log (x - 3) = \log 27$

(g) $\log_3 x + 2 \cdot \log_x 3 = 3$

(h) $\log_2 (x + 2) - \log_2 x = 3$

(i) $e^{\sin x} = 1$

(j) $\log_4 (3x - 5) = 0$.

(6) Solve the following :

(a) $3^{x-1} + 3^x - 12 = 0$

(b) $4^x + 1 = 2^{x+1}$

(c) $\frac{1}{4} \log_5 x = \log_x 5$

(d) $\log_5 x + 1 = 2 \log_x 5$

(e) $\log_2 x - \log_4 x = 2$

(f) $\log_2 [\log_2 (\log_3 x)] = 1$

(g) $\log_{49} [\log_2 (5x - 2)] = \frac{1}{2}$.

(7) Evaluate : $\frac{1}{\log_8 2} + \frac{1}{\log_4 2}$.

(8) Evaluate : $2 \cdot \log_3 5 + \log_3 40 - 3 \log_3 10$.

(9) Given $\log (x^3 y^2) = 9$ and $\log \left(\frac{x}{y}\right) = 2$, find $\log x$, $\log y$.

(10) Given that $\log_{27} x = a$ and $\log_9 y = b$, find $\log_3 \left(\frac{x}{y}\right)$.

(11) Given $a = 3^p$ and $b = 3^q$ find $\log_9 a + \log_{27} b$ in terms of p and q.

(12) Obtain y in terms of x if $\log_{10} (y - 1) = 3 \log_{10} x + 2$.

(13) Given that $2 \log_2 (x + y) = 3 + \log_2 x + \log_2 y$.
Show that $x^2 + y^2 = 6xy$.

(14) Prove that : $\frac{1}{\log_b a} + \frac{1}{\log_c a} = \frac{1}{\log_{bc} a}$.

(15) Prove that : $\frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc} = 1$.

(16) Prove that : $\frac{1}{\log_a bc + 1} + \frac{1}{\log_b ac + 1} + \frac{1}{\log_c ab + 1} = 1$.

(17) Prove that : $\log \left(\frac{p}{q}\right) + \log \left(\frac{q}{r}\right) + \log \left(\frac{r}{p}\right) = 0$.

(18) If $\log_r p = q$ and $\log_q r = p$ prove that $\log_q p = pq$.

(19) Prove that : $\frac{1}{\log_{bc} a + 1} + \frac{1}{\log_{ac} b + 1} + \frac{1}{\log_{ab} c + 1} = 2$

ANSWERS

(1) (a) $\log_2 8 = 3$ (b) $\log_8 1 = 0$
(c) $\log_{10} 0.1 = -1$ (d) $\log_4 \frac{1}{16} = -2$
(e) $\log_{27} \frac{1}{3} = -\frac{1}{3}$.

(2) (a) $p = q^r$ (b) $2^5 = 32$
(c) $10^{-2} = 0.01$ (d) $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$
(e) $(27)^{2/3} = 9$.

(3) (a) $\frac{7}{2}$ (b) $\frac{1}{2}$
(c) -8 (d) -3
(e) $\frac{3}{4}$ (f) $\frac{4}{3}$
(g) $\frac{3}{4}$ (h) 1
(i) $\frac{7}{4}$.

(4) (a) $\log (72)$ (b) $\log (100)$
(c) $\log_{10} (2000)$ (d) $\log \left(\frac{18}{35}\right)$.

(5) (a) $x = \frac{9}{4}$ (b) $x = \pm 3$
(c) $x = 4$ (d) $x = 2$
(e) $x = -12$ or 6 (f) $x = 6$
(g) $x = 3$ or 9 (h) $x = \frac{2}{7}$
(i) $x = 0$ (j) $x = 2$.

(6) (a) $x = 2$ (b) $x = 0$

(c) $x = 25$ or $\frac{1}{25}$ (d) $x = 5$ or $\frac{1}{25}$

(e) $x = 16$ (f) $x = 81$

(g) $x = 26$.

(7) 5

(8) 0

(9) $\frac{13}{5}, \frac{3}{5}$

(10) $3a - 2b$

(11) $\frac{p}{2} + \frac{q}{3}$

(12) $y = 1 + 100x^3$.

●●● PROBLEMS OF BOARD PAPERS ●●●

(1) Simplify : $\frac{1}{\log_5 10} + \frac{1}{\log_{20} 10}$. [W'09, Marks]

Ans. Solved Problem (8) on Page 1.13.

(2) Find the value using rule of Logarithm : [W'09, Marks]

$$2\log\left(\frac{3}{4}\right) + \log\left(13\frac{1}{3}\right) - \log\left(7\frac{1}{2}\right).$$

Ans. Solved Problem (11) on Page 1.14.

(3) If $\log_3 (x + 6) = 2$, find 'x'. [S'10, Marks]

Ans. Given $\log_3 (x + 6) = 2$

$\therefore$ $x + 6 = 3^2$ ... Converting into exponential form

$\therefore$ $x + 6 = 9$

$\therefore$ $x = 9 - 6 = 3$.

(4) Prove that : $\frac{1}{\log_3 6} + \frac{1}{\log_8 6} + \frac{1}{\log_9 6} = 3$. [S'10, Marks]

Ans. L.H.S. $= \frac{1}{\log_3 6} + \frac{1}{\log_8 6} + \frac{1}{\log_9 6}$

$= \log_6 3 + \log_6 8 + \log_6 9$ ... By rule of change of base

$= \log_6 (3 \times 8 \times 9)$

$= \log_6 216 = \log_6 6^3 = 3\log_6 6 = 3 \times 1 = 3$

= R.H.S.

Chapter 2

PARTIAL FRACTIONS

2.1 INTRODUCTION

Consider the addition of the simple fractions as shown below :

(i) $\frac{1}{x+2} + \frac{1}{x-1} = \frac{2x+1}{(x-1)(x+2)} = \frac{2x+1}{x^2+x-2}$

(ii) $\frac{1}{x} + \frac{2}{x-1} - \frac{3}{x+1} = \frac{5x-1}{x(x-1)(x+1)} = \frac{5x-1}{x^3-x}$

In these examples, if by some process, we are able to break up $\frac{2x+1}{x^2+x-2}$ as $\frac{1}{x+2} + \frac{1}{x-1}$ and $\frac{5x-1}{x^3-x}$ as $\frac{1}{x} + \frac{2}{x-1} - \frac{3}{x+1}$, then the simple fractions into which a single original fraction is splitted are called its corresponding **Partial Fractions**.

Any fraction of the form $\frac{P(x)}{Q(x)}$, where P (x) and Q (x) are the polynomials in x, is known as a rational fraction, such rational algebraic expressions are of two types.

(a) **Proper fractions :** In the rational fraction $\frac{P(x)}{Q(x)}$, if the numerator i.e. P (x) has the degree smaller than that of the denominator i.e. Q (x), then $\frac{P(x)}{Q(x)}$ is called proper fraction. For example,

$$\frac{x}{x^2+2x+3}, \frac{x+1}{(x-2)(x+3)}, \frac{x-5}{x^3+x^2-6x} \text{ ... are proper fractions.}$$

To obtain the partial fractions of the proper fraction $\frac{P(x)}{Q(x)}$, it should be noted that Q (x) is factorisable into linear or irreducible quadratic factors.

(b) **Improper fractions :** If the degree of the polynomial P (x) (i.e. numerator) is greater than or equal to that of the polynomial Q (x) (i.e. denominator), then the rational fraction $\frac{P(x)}{Q(x)}$ is called improper fraction. For example,

$$\frac{x^2+1}{x^2-1}, \frac{x^3}{x^2-1} \text{ ... are improper fractions.}$$

2.2 IMPORTANT STEPS REGARDING PARTIAL FRACTIONS

There are two important steps to be taken into account before resolving a given rational fraction in group of partial fractions.

Step - I : Observe the fraction carefully and decide whether it is a proper fraction or improper fraction. If $\frac{P(x)}{Q(x)}$ is improper fraction, we divide P (x) by Q (x) until the degree of the remainder is less than the Q (x). Then express it as a sum of a polynomial and a proper fraction. Thus,

Rational Fraction, $\frac{P(x)}{Q(x)}$ = Polynomial + Proper Fraction

OR

$= \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$

For instance, $\frac{x^3}{x^2-1}$ is an improper fraction. Thus, we have

x ← Quotient
∴ $x^2 - 1$) x^3
↑ $x^3 - x$
− +
Divisor x → Remainder

$$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$$ (Proper frac...

Here, $\frac{x}{x^2-1}$ can be resolved into partial fraction...

Step - II : The denominator, if possible, is factorized into simpliest factors. Repeated factors should always be combined together at one place. Study the following cases carefully.

(a) $\frac{1}{(x+1)(x^3+1)} = \frac{1}{(x+1)(x+1)(x^2-x+1)} = \frac{1}{(x+1)^2(x^2-x+1)}$

(b) $\frac{1}{(x-1)(x^2-1)} = \frac{1}{(x-1)(x-1)(x+1)} = \frac{1}{(x-1)^2(x+1)}$

(c) $\frac{1}{x^3-x^2} = \frac{1}{x^2(x-1)}$

We now intend to put a given fraction into the sum or difference of fractions. This process is known as finding the partial fractions of the original fraction. Then each fraction is called the partial fraction of the original fraction.

2.3 DIFFERENT CASES OF PARTIAL FRACTIONS

While obtaining partial fractions, four cases arise accordingly as the nature of the denominator in the fraction.

Case - I : When the denominator i.e. Q (x) has non-repeated linear factors or it can be factorized in distinct linear factors.

In this case, the partial fractions of the rational fraction will be of the form :

$$\frac{P(x)}{Q(x)} = \frac{A}{1^{st}\text{ factor}} + \frac{B}{2^{nd}\text{ factor}} + \frac{C}{3^{rd}\text{ factor}} + \dots$$

Now, to fine constants A, B, C, we have the following steps :

(a) Multiply both sides by L.C.M. of denominators on R.H.S.

(b) This gives on equating numerators of both sides, an identity which is true for all values of x.

(c) Find A, B, C by substituting proper values of x in the relation obtained from above step (b).

(d) Substituting the values of A, B, C, in the above expression for $\frac{P(x)}{Q(x)}$, we get the required partial fractions.

.... ILLUSTRATIVE EXAMPLES ....

(1) Resolve into partial fractions : $\frac{x+1}{(x+3)(x-2)}$

Solution :

Let $\frac{x+1}{(x+3)(x-2)} = \frac{A}{x-2} + \frac{B}{x+3}$

Multiplying both sides by (x + 3) (x − 2) i.e. L.C.M. of denominators on R.H.S., we get

$x + 1 = A(x+3) + B(x-2)$ which is true for all values of x.

Putting x = 2 in above relation, we get

∴ $2 + 1 = A(2+3) + 0$

∴ $3 = 5A$ ∴ $\boxed{A = \frac{3}{5}}$

Putting x = −3,

∴ $-3 + 1 = 0 + B(-3-2)$

∴ $-2 = -5B$ ∴ $\boxed{B = \frac{2}{5}}$

Thus, the required partial fractions are :

$$\frac{x+1}{(x+3)(x-2)} = \frac{\frac{3}{5}}{x-2} + \frac{\frac{2}{5}}{x+3}$$

$$= \frac{1}{5}\left[\frac{3}{x-2} + \frac{2}{x+3}\right]$$

(2) Resolve into partial fractions : $\frac{x^2+5x+7}{(x-1)(x+2)(x+4)}$

olution :

et $\frac{x^2+5x+7}{(x-1)(x+2)(x+4)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+4}$

Multiplying both sides by (x − 1) (x + 2) (x + 4), we get,

$x^2 + 5x + 7 = A(x+2)(x+4) + B(x-1)(x+4) + C(x-1)(x+2)$ which is true for all values of x.

utting x = 1,

$1 + 5 + 7 = A(1+2)(1+4) + 0 + 0$

$13 = A(3)(5)$

$13 = 15A$ ∴ $\boxed{A = \frac{13}{15}}$

Putting $x = -2$,

$\therefore \quad (-2)^2 + 5(-2) + 7 = 0 + B(-2-1)(-2+4) + 0$

$\therefore \quad 4 - 10 + 7 = B(-3)(2)$

$\therefore \quad 1 = -6B \qquad \therefore \boxed{B = \frac{-1}{6}}$

Putting $x = -4$,

$\therefore \quad (-4)^2 + 5(-4) + 7 = 0 + 0 + C(-4-1)(-4+2)$

$\therefore \quad 16 - 20 + 7 = C(-5)(-2)$

$\therefore \quad 3 = 10C \qquad \therefore \boxed{C = \frac{3}{10}}$

Thus, the required partial fractions are :

$$\frac{x^2 + 5x + 7}{(x-1)(x+2)(x+4)} = \frac{\frac{13}{15}}{x-1} + \frac{\frac{-1}{6}}{x+2} + \frac{\frac{3}{10}}{x+4}$$

$$= \frac{13}{15(x-1)} - \frac{1}{6(x+2)} + \frac{3}{10(x+4)}$$

(3) Resolve into partial fractions : $\dfrac{x}{x^2 + x - 2}$

Solution : Note that in this case $Q(x) = x^2 + x - 2$ is factorizable into distinct linear factors.

$$\therefore \quad \frac{x}{x^2 + x - 2} = \frac{x}{(x+2)(x-1)}$$

Now, let $\dfrac{x}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$

Multiplying both sides by $(x-1)(x+2)$, we get

$\therefore \quad x = A(x+2) + B(x-1)$

which is true for all values of x.

Putting $x = 1$,

$\therefore \quad 1 = A(1+2) + 0$

$\therefore \quad 1 = 3A \qquad \therefore \boxed{A = \frac{1}{3}}$

Putting $x = -2$,

$\therefore \quad -2 = 0 + B(-2-1)$

$\therefore \quad -2 = -3B \qquad \therefore \boxed{B = \frac{2}{3}}$

Thus, the required partial fractions are :

$$\frac{x}{x^2 + x - 2} = \frac{\frac{1}{3}}{x-1} + \frac{\frac{2}{3}}{x+2} = \frac{1}{3}\left[\frac{1}{x-1} + \frac{2}{x+2}\right]$$

(4) Resolve into partial fractions : $\dfrac{x-5}{x^3 + x^2 - 6x}$

Solution : Note that the denominator of the rational fraction is factorizable into distinct linear factors.

$$\therefore \quad \frac{x-5}{x^3 + x^2 - 6x} = \frac{x-5}{x(x^2 + x - 6)} = \frac{x-5}{x(x+3)(x-2)}$$

Now, let $\dfrac{x-5}{x(x-2)(x+3)} = \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+3}$

Multiplying both sides by $x(x-2)(x+3)$, we get

$\therefore \quad (x-5) = A(x-2)(x+3) + B \cdot x(x+3) + C \cdot x(x-2)$ which is true for all values of x.

Putting $x = 0$,

$0 - 5 = A(0-2)(0+3) + 0 + 0$

$\therefore \quad -5 = A(-2)(3)$

$\therefore \quad -5 = -6A \qquad \therefore \boxed{A = \frac{5}{6}}$

Putting $x = 2$,

$2 - 5 = 0 + B(2)(2+3) + 0$

$\therefore \quad -3 = B(2)(5)$

$\therefore \quad -3 = 10B \qquad \therefore \boxed{B = \frac{-3}{10}}$

Putting $x = -3$,

$-3 - 5 = 0 + 0 + C(-3)(-3-2)$

$\therefore \quad -8 = C(-3)(-5)$

$\therefore \quad -8 = 15C \qquad \therefore \boxed{C = \frac{-8}{15}}$

Thus, the required partial fractions are :

$$\frac{x-5}{x^3 + x^2 - 6x} = \frac{\frac{5}{6}}{x} + \frac{\frac{-3}{10}}{x-2} + \frac{\frac{-8}{15}}{x+3}$$

$$= \frac{5}{6x} - \frac{3}{10(x-2)} - \frac{8}{15(x+3)}$$

(5) Resolve into partial fractions : $\dfrac{x^2 + 1}{(x^2 + 2)(x^2 + 3)}$

Solution : Observe the given fraction carefully. Each factor of the fraction contains x^2 term. This type the fraction may be resolved into partial fractions replacing x^2 by t. The method is as explained in the example.

Put $x^2 = t$,

$$\therefore \quad \frac{x^2 + 1}{(x^2 + 2)(x^2 + 3)} = \frac{t+1}{(t+2)(t+3)}$$

Now, let $\dfrac{t+1}{(t+2)(t+3)} = \dfrac{A}{t+2} + \dfrac{B}{t+3}$

Multiplying both sides by $(t+2)(t+3)$, we get,

$\therefore \quad (t+1) = A(t+3) + B(t+2)$ which is true for all values of t.

Putting $t = -2$,

$-2 + 1 = A(-2+3) + 0$

$\therefore \quad -1 = A \qquad \therefore \boxed{A = -1}$

Putting $t = -3$,

$$-3 + 1 = 0 + B(-3 + 2)$$

$\therefore$ $-2 = -B$ $\quad\therefore$ $\boxed{B = 2}$

Thus, the required partial fractions are :

$$\frac{x^2 + 1}{(x^2 + 2)(x^2 + 3)} = \frac{-1}{x^2 + 2} + \frac{2}{x^2 + 3}$$

(6) **Resolve into partial fractions :** $\frac{x^3 + x}{x^2 - 4}$

Solution : This fraction is an improper as the degree of the numerator is greater than the denominator. To convert it into proper fraction, divide the numerator by the denominator by actual division.

Thus, $\frac{x^3 + x}{x^2 - 4} = x + \frac{5x}{x^2 - 4}$ ... (1)

(Proper fraction)

Divisor $x^2 - 4$	x ← Quotient
	$x^3 + x$
	$x^3 - 4x$ (− +)
	$5x$ ← Remainder

Now, let $\frac{5x}{(x^2 - 4)} = \frac{5x}{(x - 2)(x + 2)} = \frac{A}{x - 2} + \frac{B}{x + 2}$

Multiplying both sides by $(x - 2)(x + 2)$, we get

$\therefore$ $5x = A(x + 2) + B(x - 2)$ which is true for all values of x.

Putting $x = 2$,

$$5(2) = A(2 + 2) + 0$$

$\therefore$ $10 = 4A$ $\quad\therefore$ $\boxed{A = \frac{10}{4} = \frac{5}{2}}$

Putting $x = -2$,

$$5(-2) = 0 + B(-2 - 2)$$

$\therefore$ $-10 = -4B$

$\therefore$ $10 = 4B$ $\quad\therefore$ $\boxed{B = \frac{10}{4} = \frac{5}{2}}$

Then, from (1) above, the required partial fractions are :

$$\frac{x^3 + x}{x^2 - 4} = x + \frac{\frac{5}{2}}{x - 2} + \frac{\frac{5}{2}}{x + 2} = x + \frac{5}{2}\left[\frac{1}{x - 2} + \frac{1}{x + 2}\right]$$

(7) **Resolve into partial fractions :** $\frac{x^2 + 1}{x^2 - 1}$

Solution : This fraction is an improper as the degree of the numerator is equal to that of denominator. To convert it into proper fraction, divide $x^2 + 1$ by $x^2 - 1$ by actual division.

Thus, $\frac{x^2 + 1}{x^2 - 1} = 1 + \frac{2}{x^2 - 1}$ ... (1)

(Proper fraction)

Divisor $x^2 - 1$	1 ← Quotient
	$x^2 + 1$
	$x^2 - 1$ (− +)
	2 ← Remainder

Now, let $\frac{2}{(x^2 - 1)} = \frac{2}{(x - 1)(x + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1}$

Multiplying both sides by $(x - 1)(x + 1)$, we get

$\therefore$ $2 = A(x + 1) + B(x - 1)$ which is true for all values of x.

Putting $x = 1$,

$$2 = A(1 + 1) + 0$$

$\therefore$ $2 = 2A$ $\quad\therefore$ $\boxed{A = 1}$

Putting $x = -1$,

$$2 = 0 + B(-1 - 1)$$

$\therefore$ $2 = -2B$ $\quad\therefore$ $\boxed{B = -1}$

Then, from (1) above, the required partial fractions are :

$$\frac{x^2 + 1}{x^2 - 1} = 1 + \frac{1}{x - 1} + \frac{-1}{x + 1}$$

$$= 1 + \frac{1}{x - 1} - \frac{1}{x + 1}$$

Case - II : When the denominator i.e. Q (x) has repeated linear factors say $(ax + b)^r$.

In this case, the partial fractions are of the form :

$$\frac{P(x)}{(ax + b)^r} = \frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \dots + \frac{A_r}{(ax + b)^r}$$

To find $A_1, A_2, \dots A_r$, multiply both sides by L.C.M. of denominators on R.H.S. This gives an identity in x. Substituting proper values of x, we fine $A_1, A_2, \dots$. The following examples illustrate the method.

.... ILLUSTRATIVE EXAMPLES ....

(1) **Resolve into partial fractions :** $\frac{2x + 1}{x^2(x + 1)}$

Solution : In this fraction, partial fractions are of the form

$$\frac{2x + 1}{x^2(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1}$$

Multiplying both sides by $x^2(x + 1)$ i.e. L.C.M., of denominators on R.H.S., we get

$\therefore$ $2x + 1 = A \cdot x(x + 1) + B(x + 1) + C \cdot x^2$ which is true for all values of x.

Putting $x = 0$,

$\therefore \quad 0 + 1 = 0 + B(0 + 1) + 0$

$\therefore \quad 1 = B \qquad \therefore \boxed{B = 1}$

Putting $x = -1$,

$\therefore \quad 2(-1) + 1 = 0 + 0 + C(-1)^2$

$\therefore \quad -2 + 1 = C(1)$

$\therefore \quad -1 = C \qquad \therefore \boxed{C = -1}$

Putting $x = 1, B = 1, C = -1$,

$2(1) + 1 = A(1)(1 + 1) + (1)(1 + 1) + (-1)(1)^2$

$\therefore \quad 3 = A.(2) + 2 - 1$

$\therefore \quad 3 = 2A + 1$

$\therefore \quad 2A = 2 \qquad \therefore \boxed{A = 1}$

Thus, the required partial fractions are :

$$\frac{2x + 1}{x^2(x + 1)} = \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x + 1}$$

(2) Resolve into partial fractions : $\dfrac{3x + 2}{(x + 1)(x^2 - 1)}$

Solution : Note that the given fraction has repeated linear factors in the denominator as shown be[low]

$$\therefore \quad \frac{3x + 2}{(x + 1)(x^2 - 1)} = \frac{3x + 2}{(x + 1)(x + 1)(x - 1)} = \frac{3x + 2}{(x - 1)(x + 1)^2}$$

Now, let $\dfrac{3x + 2}{(x - 1)(x + 1)^2} = \dfrac{A}{x - 1} + \dfrac{B}{x + 1} + \dfrac{C}{(x + 1)^2}$

Multiplying both sides by $(x - 1)(x + 1)^2$, we get,

$\therefore \quad 3x + 2 = A(x + 1)^2 + B(x - 1)(x + 1) + C(x - 1)$ which is true fo[r all] values of x.

Putting $x = 1$,

$\therefore \quad 3(1) + 2 = A(1 + 1)^2 + 0 + 0$

$3 + 2 = 4A$

$\therefore \quad 5 = 4A \qquad \therefore \boxed{A = \frac{5}{4}}$

Putting $x = -1$,

$\therefore \quad 3(-1) + 2 = 0 + 0 + C(-1 - 1)$

$\therefore \quad -3 + 2 = C(-2)$

$-1 = -2C \qquad \therefore \boxed{C = \frac{1}{2}}$

Putting $x = 0, A = \frac{5}{4}, C = \frac{1}{2}$,

$\therefore \quad 3(0) + 2 = \frac{5}{4}(0 + 1)^2 + B(0 - 1)(0 + 1) + \frac{1}{2}(0 - 1)$

$0 + 2 = \frac{5}{4}(1) + B(-1)(1) - \frac{1}{2}$

$\therefore \quad 2 = \frac{5}{4} - B - \frac{1}{2}$

$\therefore \quad B = \frac{5}{4} - \frac{1}{2} - 2$

$\therefore \quad B = \frac{5 - 2 - 8}{4} \qquad \therefore \boxed{B = \frac{-5}{4}}$

Thus, the required partial fractions are :

$$\frac{3x + 2}{(x + 1)(x^2 - 1)} = \frac{\frac{5}{4}}{x - 1} + \frac{\frac{-5}{4}}{x + 1} + \frac{\frac{1}{2}}{(x + 1)^2}$$

$$= \frac{5}{4(x - 1)} - \frac{5}{4(x + 1)} + \frac{1}{2(x + 1)^2}$$

(3) Resolve into partial fractions : $\dfrac{x^2 + x + 1}{(x - 2)(x^2 - 4)}$

Solution : Here $\dfrac{x^2 + x + 1}{(x - 2)(x^2 - 4)} = \dfrac{x^2 + x + 1}{(x - 2)(x - 2)(x + 2)} = \dfrac{x^2 + x + 1}{(x + 2)(x - 2)^2}$

Let $\dfrac{x^2 + x + 1}{(x + 2)(x - 2)^2} = \dfrac{A}{x + 2} + \dfrac{B}{x - 2} + \dfrac{C}{(x - 2)^2}$

Multiplying both sides by $(x + 2)(x - 2)^2$, we get,

$\therefore \quad x^2 + x + 1 = A(x - 2)^2 + B(x + 2)(x - 2) + C(x + 2)$ which is true for all values of x.

Putting $x = 2$,

$(2)^2 + 2 + 1 = 0 + 0 + C(2 + 2)$

$\therefore \quad 4 + 2 + 1 = C(4)$

$\therefore \quad 7 = 4C \qquad \therefore \boxed{C = \frac{7}{4}}$

Putting $x = -2$,

$(-2)^2 + (-2) + 1 = A(-2 - 2)^2 + 0 + 0$

$\therefore \quad 4 - 2 + 1 = A(-4)^2$

$\therefore \quad 3 = 16A \qquad \therefore \boxed{A = \frac{3}{16}}$

Putting $x = 0, A = \frac{3}{16}, C = \frac{7}{4}$,

$0 + 0 + 1 = \frac{3}{16}(0 - 2)^2 + B(0 + 2)(0 - 2) + \frac{7}{4}(0 + 2)$

$\therefore \quad 1 = \frac{3}{16}(4) + B(2)(-2) + \frac{7}{4}(2)$

$\therefore \quad 1 = \frac{3}{4} - 4B + \frac{7}{2}$

$\therefore \quad 4B = \frac{3}{4} + \frac{7}{2} - 1$

$\therefore \quad 4B = \frac{3 + 14 - 4}{4} = \frac{13}{4} \qquad \therefore \boxed{B = \frac{13}{16}}$

Thus, the required partial fractions are :

$$\frac{x^2 + x + 1}{(x - 2)(x^2 - 4)} = \frac{\frac{3}{16}}{x + 2} + \frac{\frac{13}{16}}{x - 2} + \frac{\frac{7}{4}}{(x - 2)^2}$$

$$= \frac{1}{16}\left[\frac{3}{x + 2} + \frac{13}{x - 2} + \frac{28}{(x - 2)^2}\right]$$

Case - III : When the denominator has non-repeated irreducible quadratic factor :

When the denominator i.e. Q (x) can be resolved into real factors, some of which are irr[educible] quadratic factors, say of the type $(ax^2 + bx + c)$, which cannot be further resolved into real linear [factors], then corresponding to each irreducible quadratic factor $ax^2 + bx + c$, there occurs a partial fracti[on of the] form $\frac{Ax + B}{ax^2 + bx + c}$, where A and B are constants. The following examples illustrate the method.

.... ILLUSTRATIVE EXAMPLES ....

(1) Resolve into partial fractions : $\frac{x^2 + 23x}{(x + 3)(x^2 + 1)}$

Solution : Here in denominator, $(x^2 + 1)$ is an irreducible quadratic factor.

Let $\frac{x^2 + 23x}{(x + 3)(x^2 + 1)} = \frac{A}{x + 3} + \frac{Bx + C}{x^2 + 1}$

Multiplying both sides by $(x + 3)(x^2 + 1)$, we get,

$\therefore \quad x^2 + 23x = A(x^2 + 1) + (Bx + C)(x + 3)$ which is true for all values of x.

Putting $x = -3$,

$(-3)^2 + 23(-3) = A\{(-3)^2 + 1\} + 0$

$\therefore \quad 9 - 69 = A(9 + 1)$

$\therefore \quad -60 = 10A \qquad \therefore \boxed{A = -6}$

Putting $x = 0, A = -6$,

$0 + 0 = (-6)(0 + 1) + (0 + C)(0 + 3)$

$\therefore \quad 0 = -6 + 3C$

$\therefore \quad 3C = 6 \qquad \therefore \boxed{C = 2}$

Putting $x = 1, A = -6, C = 2$,

$(1)^2 + 23(1) = (-6)(1 + 1) + (B + 2)(1 + 3)$

$\therefore \quad 1 + 23 = (-6)(2) + (B + 2)(4)$

$\therefore \quad 24 = -12 + 4B + 8$

$\therefore \quad 24 + 12 - 8 = 4B$

$\therefore \quad 4B = 28 \qquad \therefore \boxed{B = 7}$

Thus, the required partial fractions are :

$$\frac{x^2 + 23x}{(x + 3)(x^2 + 1)} = \frac{-6}{x + 3} + \frac{7x + 2}{x^2 + 1}$$

(2) Resolve into partial fractions : $\frac{x^2 + 1}{x^3 + 1}$

Solution : Here note that Q (x) is factorisable into a combination of a linear factor and an irred[ucible] *quadratic factor.*

$$\frac{x^2 + 1}{x^3 + 1} = \frac{x^2 + 1}{(x + 1)(x^2 - x + 1)}$$

Let, $\frac{x^2 + 1}{(x + 1)(x^2 - x + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1}$

Multiplying both sides by $(x + 1)(x^2 - x + 1)$, we get

$\therefore \quad x^2 + 1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$ which is true for all values of x.

Putting $x = -1$,

$(-1)^2 + 1 = A\{(-1)^2 - (-1) + 1\} + \{B(-1) + C\}(-1 + 1)$

$\therefore \quad 1 + 1 = A(1 + 1 + 1) + 0$

$\therefore \quad 2 = 3A \qquad \therefore \boxed{A = \frac{2}{3}}$

Putting $x = 0, A = \frac{2}{3}$

$0 + 1 = \frac{2}{3}(0 - 0 + 1) + (0 + C)(0 + 1)$

$\therefore \quad 1 = \frac{2}{3} + C$

$\therefore \quad C = 1 - \frac{2}{3} = \frac{3 - 2}{3} = \frac{1}{3} \qquad \therefore \boxed{C = \frac{1}{3}}$

Putting $x = 1, A = \frac{2}{3}, C = \frac{1}{3}$

$(1)^2 + 1 = \left(\frac{2}{3}\right)\{(1)^2 - (1) + 1\} + \left\{B(1) + \frac{1}{3}\right\}(1 + 1)$

$\therefore \quad 1 + 1 = \frac{2}{3}(1 - 1 + 1) + \left(B + \frac{1}{3}\right)(2)$

$\therefore \quad 2 = \frac{2}{3} + 2B + \frac{2}{3}$

$\therefore \quad 2B = 2 - \frac{2}{3} - \frac{2}{3} = \frac{6 - 2 - 2}{3} = \frac{2}{3}$

$\therefore \quad 2B = \frac{2}{3} \qquad \therefore \boxed{B = \frac{1}{3}}$

Thus, the required partial fractions are :

$$\frac{x^2 + 1}{x^3 + 1} = \frac{\frac{2}{3}}{x + 1} + \frac{\frac{1}{3}x + \frac{1}{3}}{x^2 - x + 1} = \frac{1}{3}\left[\frac{2}{x + 1} + \frac{x + 1}{x^2 - x + 1}\right]$$

(3) Resolve into partial fractions : $\frac{x + 2}{(x - 1)(x^2 + x + 1)}$

Solution : In this case, the factor $x^2 + x + 1$ is an irreducible quadratic expression.

Let $\frac{x + 2}{(x - 1)(x^2 + x + 1)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + x + 1}$

Multiplying both sides by $(x - 1)(x^2 + x + 1)$, we get

$\therefore \quad x + 2 = A(x^2 + x + 1) + (Bx + C)(x - 1)$ which is true for all values of x.

Putting $x = 1$,

$$1 + 2 = A\{(1)^2 + 1 + 1\} + \{B(1) + C\}(1 - 1)$$

$\therefore$ $3 = A(1 + 1 + 1) + 0$

$\therefore$ $3 = 3A$ $\therefore$ $\boxed{A = 1}$

Putting $x = 0$, $A = 1$,

$$0 + 2 = (1)\{0 + 0 + 1\} + \{0 + C\}(0 - 1)$$

$\therefore$ $2 = 1 - C$

$\therefore$ $C = 1 - 2$ $\therefore$ $\boxed{C = -1}$

Putting $x = -1$, $A = 1$, $C = -1$,

$$-1 + 2 = (1)\{(-1)^2 + (-1) + 1\} + \{B(-1) + (-1)\}(-1 - 1)$$

$\therefore$ $1 = \{1 - 1 + 1\} + \{-B - 1\}(-2)$

$\therefore$ $1 = 1 + 2B + 2$

$\therefore$ $2B = -2$ $\therefore$ $\boxed{B = -1}$

Thus, the required partial fractions are :

$$\frac{x + 2}{(x - 1)(x^2 + x + 1)} = \frac{1}{x - 1} + \frac{(-1)x + (-1)}{x^2 + x + 1} = \frac{1}{x - 1} - \frac{x + 1}{x^2 + x + 1}$$

(4) Resolve into partial fractions : $\dfrac{x^2 - 2x + 3}{(x^3 + x)}$

Solution : Here $\dfrac{x^2 - 2x + 3}{x^3 + x} = \dfrac{x^2 - 2x + 3}{x(x^2 + 1)}$

Let $\dfrac{x^2 - 2x + 3}{x(x^2 + 1)} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 1}$

Multiplying both sides by $x(x^2 + 1)$, we get

$\therefore$ $x^2 - 2x + 3 = A(x^2 + 1) + (Bx + C)x$ which is true for all values of x.

Putting $x = 0$,

$\therefore$ $0 - 0 + 3 = A(0 + 1) + (0 + C)(0)$

$\therefore$ $3 = A$ $\therefore$ $\boxed{A = 3}$

Putting $x = 1$, $A = 3$,

$\therefore$ $(1)^2 - 2(1) + 3 = (3)(1 + 1) + (B + C)(1)$

$\therefore$ $1 - 2 + 3 = (3)(2) + B + C$

$\therefore$ $2 = 6 + B + C$

$\therefore$ $B + C = -4$ ... (1)

Putting $x = -1$, $A = 3$

$$(-1)^2 - 2(-1) + 3 = 3(1 + 1) + (-B + C)(-1)$$

$\therefore$ $1 + 2 + 3 = 3(2) + B - C$

$\therefore$ $6 = 6 + B - C$

$\therefore$ $B - C = 0$ ... (2)

Adding (1) and (2) to solve for B.

$\therefore$ $2B = -4$ $\therefore$ $\boxed{B = -2}$

From (1) when $B = -2$, we get

$\therefore$ $-2 + C = -4$ $\therefore$ $\boxed{C = -2}$

Thus, the required partial fractions are :

$$\frac{x^2 - 2x + 3}{x^3 + x} = \frac{3}{x} + \frac{-2x - 2}{x^2 + 1} = \frac{3}{x} - \frac{2x + 2}{x^2 + 1}$$

Case - IV : When Q (x) has repeated irreducible quadratic factors :

The method is illustrated in the following example :

.... ILLUSTRATIVE EXAMPLE ....

(1) Resolve into partial fractions : $\dfrac{x^2 + x + 5}{(x^2 + 1)^2}$

Solution :

Let $\dfrac{x^2 + x + 5}{(x^2 + 1)^2} = \dfrac{Ax + B}{x^2 + 1} + \dfrac{Cx + D}{(x^2 + 1)^2}$

Multiplying both sides by $(x^2 + 1)^2$, we get,

$\therefore$ $x^2 + x + 5 = (Ax + B)(x^2 + 1) + Cx + D$ simplifying R.H.S., we get

$\therefore$ $0x^3 + x^2 + x + 5 = Ax^3 + Bx^2 + (A + C)x + (B + D)$

Now, equating coefficients of respective terms on both sides, we get

$\therefore$ $\boxed{A = 0}$, $\boxed{B = 1}$, $A + C = 1$, $B + D = 5$

Putting $A = 0$, in $A + C = 1$

$\therefore$ $0 + C = 1$ $\therefore$ $\boxed{C = 1}$

Putting $B = 1$ in $B + D = 5$

$\therefore$ $1 + D = 5$ $\therefore$ $\boxed{D = 4}$

Thus, the required partial fractions are :

$$\frac{x^2 + x + 5}{(x^2 + 1)^2} = \frac{1}{x^2 + 1} + \frac{x + 4}{(x^2 + 1)^2}$$

.... EXERCISES ....

Resolve the following into partial fractions :

(1) (a) $\dfrac{1}{(x + 2)(x + 3)}$ (b) $\dfrac{x + 4}{x(x + 1)}$ (c) $\dfrac{x + 1}{(x - 2)(x + 3)}$

(d) $\dfrac{x + 18}{(x - 3)(2x + 1)}$ (e) $\dfrac{x}{(2x - 1)(x + 4)}$ (f) $\dfrac{4x^2 + 5}{(x - 2)(x + 1)(2x - 3)}$

(g) $\dfrac{x + 4}{x(x + 1)(x + 2)}$ (h) $\dfrac{3x^2 + x + 4}{(x - 2)(2x + 1)(3x + 4)}$ (i) $\dfrac{x + 6}{x(x + 1)(x + 4)}$

(2) (a) $\dfrac{1}{x^2 - x}$ (b) $\dfrac{1}{x^2 - 1}$ [W'07] (c) $\dfrac{1}{x^2 + 3x + 2}$ [S'08]

(d) $\dfrac{x}{x^2 + x - 2}$ (e) $\dfrac{5x - 1}{x^2 - 5x - 6}$ (f) $\dfrac{2x - 3}{(x^2 - 1)(2x + 3)}$

(g) $\dfrac{4x^2 + x - 1}{x^3 - x}$ (h) $\dfrac{x^2 + 1}{x(x^2 - 1)}$ [S'08]

(3) (a) $\dfrac{x^2 - 1}{(x^2 + 1)(x^2 + 2)}$ (b) $\dfrac{x^2}{(x^2 - 1)(x^2 + 2)}$ (c) $\dfrac{2x^4 + x^2 \ldots}{(x^2 + 1)(x^2 - 2) \ldots}$

Note : Put $x^2 = t$ to resolve into partial fractions.

(4) (a) $\dfrac{3x + 4}{(x + 6)(x + 2)^2}$ (b) $\dfrac{3x + 5}{(x + 3)(x + 1)^2}$ (c) $\dfrac{x^2}{(x + 1)(x + 2)^2}$

(d) $\dfrac{2x^2 + 3}{(x - 2)(x + 1)^2}$ (e) $\dfrac{x^2 - 2x - 7}{(x + 1)(x - 1)^2}$ (f) $\dfrac{3x + 2}{(x + 1)(x^2 - 1)}$

(5) (a) $\dfrac{1}{x^3 - 1}$ (b) $\dfrac{1}{x^3 + 1}$ (c) $\dfrac{2x + 1}{(x - 1)(x^2 + 1)}$

(d) $\dfrac{x - 5}{x^3 + x^2 - 5x}$ (e) $\dfrac{x^2 + x + 3}{(x + 3)(x^2 + 2)}$ (f) $\dfrac{3x^2 + 17x + 14}{x^3 - 8}$

(g) $\dfrac{x}{(x^3 - 1)(x + 2)}$ (h) $\dfrac{x^2 + 1}{(x + 1)(x^2 + 4)}$

(6) (a) $\dfrac{x^3}{x^2 - 1}$ (b) $\dfrac{x^4}{x^3 + 1}$ **[W'07, S'08]** (c) $\dfrac{x^4}{x^3 - 1}$

(7) (a) $\dfrac{2x^2 + 4x - 5}{(x^2 + 2)^2}$ (b) $\dfrac{x^3 + 3x^2 + 4}{(x^2 + 1)^2}$

.... ANSWERS ....

(1) (a) $\dfrac{1}{x + 2} - \dfrac{1}{x + 3}$ (b) $\dfrac{4}{x} - \dfrac{3}{x + 1}$

(c) $\dfrac{1}{5}\left[\dfrac{3}{x - 2} + \dfrac{2}{x + 3}\right]$ (d) $\dfrac{3}{x - 3} - \dfrac{5}{2x + 1}$

(e) $\dfrac{1}{9}\left[\dfrac{1}{2x - 1} + \dfrac{4}{x + 4}\right]$ (f) $\dfrac{7}{x - 2} + \dfrac{3}{5(x + 1)} - \dfrac{56}{5(2x - 3)}$

(g) $\dfrac{2}{x} - \dfrac{3}{x + 1} + \dfrac{1}{x + 2}$ (h) $\dfrac{1}{25}\left[\dfrac{9}{x - 2} - \dfrac{17}{2x + 1} + \dfrac{36}{3x + 4}\right]$

(i) $\dfrac{3}{2x} - \dfrac{5}{3(x + 1)} + \dfrac{1}{6(x + 4)}$

(2) (a) $\dfrac{1}{x - 1} - \dfrac{1}{x}$ (b) $\dfrac{1}{2}\left[\dfrac{1}{x - 1} - \dfrac{1}{x + 1}\right]$

(c) $\dfrac{1}{x + 1} - \dfrac{1}{x + 2}$ (d) $\dfrac{1}{3}\left[\dfrac{2}{x + 2} + \dfrac{1}{x - 1}\right]$

(e) $\dfrac{1}{7}\left[\dfrac{29}{x - 6} + \dfrac{6}{x + 1}\right]$ (f) $\dfrac{-1}{10(x - 1)} + \dfrac{5}{2(x + 1)} - \dfrac{24}{5(2x + 3)}$

(g) $\dfrac{1}{x} + \dfrac{2}{x - 1} + \dfrac{1}{x + 1}$ (h) $\dfrac{-1}{x} + \dfrac{1}{x + 1} + \dfrac{1}{x - 1}$

(3) (a) $\dfrac{3}{x^2 + 2} - \dfrac{2}{x^2 + 1}$ (b) $\dfrac{1}{3(x^2 - 1)} + \dfrac{2}{3(x^2 + 2)}$

(c) $\dfrac{-5}{3(x^2 + 1)} + \dfrac{2}{3(x^2 - 2)} + \dfrac{4}{2x^2 + 3}$

(4) (a) $\dfrac{-7}{8(x + 6)} + \dfrac{7}{8(x + 2)} - \dfrac{1}{2(x + 2)^2}$ (b) $\dfrac{-1}{x + 3} + \dfrac{1}{x + 1} + \dfrac{1}{(x + 1)^2}$

(c) $\dfrac{1}{x + 1} - \dfrac{4}{(x + 2)^2}$ (d) $\dfrac{11}{9(x - 2)} + \dfrac{7}{9(x + 1)} - \dfrac{5}{3(x + 1)^2}$

(e) $\dfrac{-1}{x + 1} + \dfrac{2}{x - 1} - \dfrac{4}{(x - 1)^2}$ (f) $\dfrac{5}{4(x - 1)} - \dfrac{5}{4(x + 1)} + \dfrac{1}{2(x + 1)^2}$

(5) (a) $\dfrac{1}{3}\left[\dfrac{1}{x - 1} - \dfrac{x + 2}{x^2 + x + 1}\right]$ (b) $\dfrac{1}{3}\left[\dfrac{1}{x + 1} - \dfrac{x - 2}{x^2 - x + 1}\right]$

(c) $\dfrac{3}{2(x - 1)} - \dfrac{3x - 1}{2(x^2 + 1)}$ (d) $\dfrac{1}{x} - \dfrac{x}{x^2 + x - 5}$

(e) $\dfrac{1}{11}\left[\dfrac{9}{x + 3} + \dfrac{2x + 5}{x^2 + 2}\right]$ (f) $\dfrac{5}{x - 2} - \dfrac{2x - 3}{x^2 + 2x + 4}$

(g) $\dfrac{1}{9(x - 1)} + \dfrac{2}{9(x + 2)} - \dfrac{x}{3(x^2 + x + 1)}$ (h) $\dfrac{2}{5(x + 1)} + \dfrac{3(x - 1)}{5(x^2 + 4)}$

(6) (a) $x + \dfrac{1}{2}\left[\dfrac{1}{x - 1} + \dfrac{1}{x + 1}\right]$ (b) $x + \dfrac{1}{3}\left[\dfrac{1}{x + 1} - \dfrac{x + 1}{x^2 - x + 1}\right]$

(c) $x + \dfrac{1}{3}\left[\dfrac{1}{x - 1} - \dfrac{x - 1}{x^2 + x + 1}\right]$

(7) (a) $\dfrac{2}{x^2 + 2} + \dfrac{4x - 9}{(x^2 + 2)^2}$ (b) $\dfrac{x + 3}{x^2 + 1} - \dfrac{x - 1}{(x^2 + 1)^2}$

●●● PROBLEMS OF BOARD PAPERS ●●●

(1) Resolve into partial fractions : $\dfrac{1}{x^2 - 1}$ **[W'07, Marks 2]**

Ans. Problem-2 (b) of Exercises on Page 2.13.

(2) Resolve into partial fractions : $\dfrac{x^2}{(x + 1)(x + 2)(x + 3)}$ **[W'07, Marks 4]**

Ans. Let $\dfrac{x^2}{(x + 1)(x + 2)(x + 3)} = \dfrac{A}{x + 1} + \dfrac{B}{x + 2} + \dfrac{C}{x + 3}$

[illegible]here

$$A = \left.\dfrac{x^2}{(x + 2)(x + 3)}\right|_{x = -1} = \dfrac{(-1)^2}{(-1 + 2)(-1 + 3)} = \dfrac{1}{(1)(2)} = \dfrac{1}{2}$$

$$B = \left.\dfrac{x^2}{(x + 1)(x + 3)}\right|_{x = -2} = \dfrac{(-2)^2}{(-2 + 1)(-2 + 3)} = \dfrac{4}{(-1)(1)} = -4$$

$$C = \left.\dfrac{x^2}{(x + 1)(x + 2)}\right|_{x = -3} = \dfrac{(-3)^2}{(-3 + 1)(-3 + 2)} = \dfrac{9}{(-2)(-1)} = \dfrac{9}{2}$$

Therefore, the required partical fractions are :

$$\dfrac{x^2}{(x + 1)(x + 2)(x + 3)} = \dfrac{\frac{1}{2}}{x + 1} + \dfrac{-4}{x + 2} + \dfrac{\frac{9}{2}}{x + 3}.$$

(3) **Resolve into partical fractions :** $\dfrac{x-2}{x^3+1}$. [W'07, M

Ans. Here $\dfrac{x-2}{x^3+1} = \dfrac{x-2}{(x+1)(x^2-x+1)}$ Note the factorization in deno

Let $\dfrac{x-2}{(x+1)(x^2-x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}$

$= \dfrac{A(x^2-x+1)+(x+1)(Bx+C)}{(x+1)(x^2-x+1)}$

This gives

$x-2 = A(x^2-x+1) + (Bx+C)(x+1)$ which is true for all x

When $x = -1$

$\therefore$ $-1-2 = A\{(-1^2)-(-1)+1\}$

$\therefore$ $-3 = A(1+1+1)$

$\therefore$ $-3 = 3A$

$\therefore$ $A = -1$

When $x = 0$

$0-2 = A(0-0+1)+(0+C)(0+1)$

$\therefore$ $-2 = (-1)(1)+C$

$\therefore$ $-2 = -1+C$

$\therefore$ $C = -1$

When $x = 1$

$1-2 = (-1)(1-1+1)+(B-1)(1+1)$

$\therefore$ $-1 = -1+2B-2$

$\therefore$ $2B = 2$

$\therefore$ $B = 1$

Therefore, the required partical fractions are :

$$\frac{x-2}{x^3+1} = \frac{-1}{x+1} + \frac{x-1}{x^2-x+1}.$$

(4) **Resolve into partial fractions :** $\dfrac{1}{x^2+3x+2}$. [S'08, Mark

Ans. Problem-2 (c) of Exercises on Page 2.13. First factorize denominator.

Let $\dfrac{1}{x^2+3x+2} = \dfrac{1}{(x+2)(x+1)} = \dfrac{A}{x+2} + \dfrac{B}{x+1}$

Complete yourself.

(5) **Resolve into partial fractions :** $\dfrac{x^2+1}{x(x^2-1)}$. [S'08, Mark

Ans. Here $\dfrac{x^2+1}{x(x^2-1)} = \dfrac{x^2+1}{x(x-1)(x+1)}$

Let $\dfrac{x^2+1}{x(x-1)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x+1}$

Find A, B, C similar to solved Problem (2) above and complete yourself.

The answer is $\dfrac{-1}{x} + \dfrac{1}{x-1} + \dfrac{1}{x+1}$.

(6) **Resolve into partial fractions :** $\dfrac{x^4}{x^3+1}$. **[S'08, Marks 4]**

Ans. Observe the fraction carefully. It is improper fraction. First make it proper by actual division.

ius,

$$\begin{array}{r|l} & x \\ x^3+1 & x^4 \\ & x^4+1 \\ & -\quad - \\ \hline & -x \end{array}$$

$\therefore$ $\dfrac{x^4}{x^3+1} = x + \dfrac{-x}{x^3+1}$

Proper

Now, $\dfrac{x}{x^3+1} = \dfrac{x}{(x+1)(x^2-x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}$

Find A, B, C as shown in above Problem (3) and complete yourself.

The answer is $x + \dfrac{1}{3}\left[\dfrac{1}{x+1} - \dfrac{x+1}{x^2-x+1}\right]$.

(7) **Resolve into partial fractions :** $\dfrac{x+4}{x(x+1)}$. **[S'09, Marks 2]**

Ans. Problem-1 (b) of Exercises on Page 2.13. Practice yourself. The answer is $\dfrac{4}{x} - \dfrac{3}{x+1}$.

(8) **Resolve into partial fractions :** $\dfrac{x^3+x}{x^2-4}$. **[S'09, Marks 4]**

Ans. Solved Problem (6) on Page 2.6. Practice yourself.

(9) **Resolve into partial fractions :** $\dfrac{2x+1}{(x-1)(x^2+1)}$. **[S'09, Marks 4]**

Ans. Let $\dfrac{2x+1}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$. Find A, B, C as usual and complete yourself.

answer is $\dfrac{1}{2}\left[\dfrac{3}{x-1} - \dfrac{3x-1}{x^2+1}\right]$.

(10) **Resolve into partial fractions :** $\dfrac{x+5}{x^2-x}$. **[W'09, Marks 2]**

Ans. Here $\dfrac{x+5}{x^2-x} = \dfrac{x+5}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1}$

This gives $x+5 = A(x-1) + B \cdot x$ which is true for all $x \in R$.

When $x = 0$, $5 = A(-1)$ $\therefore$ $A = -5$

When $x = 1$, $6 = B$ $\therefore$ $B = 6$

Therefore, the required partial fractions are :

$$\frac{x+5}{x^2-x} = \frac{-5}{x} + \frac{6}{x-1}.$$

(11) Resolve into partial fractions : $\frac{x^3 + x}{x^2 - 9}$. [W'

Ans. It is exactly similar to solved Problem (6) on Page 2.6. To convert into proper fr
$x^3 + x$ by $x^2 - 9$ as follow.

$$x^2 - 9 \overline{\smash{)}\, x^3 + x\,}\ \ x$$
$$\underline{\begin{array}{l} x^3 - 9x \\ -\quad + \end{array}}$$
$$10x$$

$$\therefore \frac{x^3 + x}{x^2 - 9} = x + \frac{10x}{x^2 - 9}$$

Now resolve $\frac{x}{x^2 - 9} = \frac{x}{(x - 3)(x + 3)} = \frac{A}{x - 3} + \frac{B}{x + 3}$

Find A, B and complete yourself. The answer is $x + 5\left[\frac{1}{x - 3} + \frac{1}{x + 3}\right]$.

(12) Resolve into partial fractions : $\frac{2x + 3}{x^2 (x - 1)}$. [W'09,

Ans. Let $\frac{2x + 3}{x^2 (x - 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1}$.

Find A, B, C as usual and complete yourself. The answer is $\frac{-5}{x} - \frac{3}{x^2} + \frac{5}{x - 1}$.

(13) Resolve into partial fractions : $\frac{x - 2}{x^2 - x}$. [S'10,

Ans. $\frac{x - 2}{x^2 - x} = \frac{x - 2}{x (x - 1)}$

Let $\frac{x - 2}{x (x - 1)} = \frac{A}{x} + \frac{B}{x - 1} = \frac{A (x - 1) + B \cdot x}{x (x - 1)}$

This gives $x - 2 = A (x - 1) + B \cdot x$ which is true for all x

When $x = 0, \ -2 = A (-1) \quad \therefore \ A = 2$

When $x = 2, \ 0 = A + B (2) \quad \therefore \ 2B = -A = -2 \quad \therefore \ B = -1$

$$\therefore \frac{x - 2}{x (x - 1)} = \frac{2}{x} - \frac{1}{x - 1}$$

These are required partial fractions.

(14) Resolve into partial fractions : $\frac{x^2 + 23x}{(x + 3)(x^2 + 1)}$. [S'10, M

Ans. Solved Problem (1) of Illustrative Examples on Page 2.10. Practice yourself.
The answer is $\frac{-6}{x + 3} + \frac{7x + 2}{x^2 + 1}$.

(15) Resolve into partial fractions : $\frac{x - 5}{x^3 + x^2 - 6x}$. [S'10, M

Ans. Solved Problem (4) of Illustrative Examples on Page 2.4. Practice yourself.
The answer is $\frac{5}{6x} - \frac{3}{10 (x - 2)} - \frac{8}{15 (x + 3)}$.

Chapter 3

DETERMINANTS

1 INTRODUCTION

The solution of the equation $ax + b = 0$ is $x = -\frac{b}{a}$, provided $a \neq 0$. Let the two equations $ax + b = 0$ d $cx + d = 0$ be satisfied by the same value of x. Hence, they are called **consistent.** From $ax + b = 0$, $= -\frac{b}{a}$ and from $cx + d = 0$, $x = -\frac{d}{c}$. Equating the values of x, we get,

$$-\frac{b}{a} = -\frac{d}{c} \quad \text{or} \quad \frac{b}{a} = \frac{d}{c} \quad \text{or} \quad ad - bc = 0$$

This expression $ad - bc$ is called **eliminant** for the equations $ax + b = 0$ and $cx + d = 0$.
This eliminant $ad - bc$ is written in a simple form as :

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

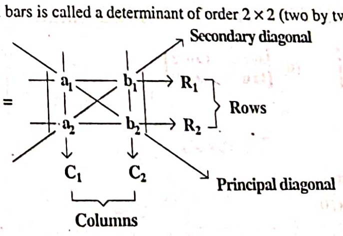
This is called determinant of order two and $ad - bc$ is called its value. Thus, **determinant is an rangement of numbers in equal number of rows (horizontal lines) and columns (vertical lines) enclosed tween two bars with some definite meaning i.e. value.**

A determinant is denoted by Δ or D or $|A|$.

2 DEFINITION OF 2 × 2 ORDER DETERMINANT

An arrangement of four numbers a_1, b_1, a_2, b_2 in two horizontal lines and two vertical lines enclosed tween two vertical bars is called a determinant of order 2 × 2 (two by two) or simply 2.

Thus, Δ or D =



Horizontal line is called a row and vertical line is called a column. a_1, b_1, a_2, b_2 are called ele... determinant.

Elements	a_1, b_1	constitute the first row (R_1)
Elements	a_2, b_2	constitute the second row (R_2)
Elements	a_1, a_2	constitute the first column (C_1)
Elements	b_1, b_2	constitute the second column (C_2)
Elements	a_1, b_2	constitute the principal diagonal
Elements	b_1, a_2	constitute the secondary diagonal.

3.2.1 Value of 2 × 2 Order Determinant

Let $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

Then its value is defined as : OR the determinant of order 2 × 2 (two by two) can be expa... **the product of the elements in the principal diagonal minus the product of the elements in the seco... diagonal.**

Thus, $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 \times b_2 - a_2 \times b_1$

$= a_1 b_2 - a_2 b_1$

For instance,

(1) $\Delta = \begin{vmatrix} 2 & -5 \\ 7 & 3 \end{vmatrix} = (2)(3) - (7)(-5)$

$= 6 - (-35) = 6 + 35 = 41$

(2) $\Delta = \begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = (\sin\theta)(\sin\theta) - (-\cos\theta)(\cos\theta)$

$= \sin^2\theta + \cos^2\theta = 1$

3.2.2 2 × 2 Determinant Equation

Any 2 × 2 order determinant of the form $\begin{vmatrix} a & b \\ c & x \end{vmatrix} = 0$, where x is unknown is called determ... equation. Note that the determinant contains atleast one unknown and is equated to zero. The value o... unknown (x) is obtained by expanding the determinant and then equating to zero.

.... ILLUSTRATIVE EXAMPLES ....

(1) Evaluate : (a) $\begin{vmatrix} 2 & -3 \\ 10 & -12 \end{vmatrix}$ (b) $\begin{vmatrix} \sec\theta & \tan\theta \\ \tan\theta & \sec\theta \end{vmatrix}$

Solution :

(a) Let $\Delta = \begin{vmatrix} 2 & -3 \\ 10 & -12 \end{vmatrix}$

Expanding the determinant, we get

$\therefore \Delta = (2)(-12) - (-3)(10)$

$= -24 + 30$

$= 6$

(b) Let $\Delta = \begin{vmatrix} \sec\theta & \tan\theta \\ \tan\theta & \sec\theta \end{vmatrix}$

Expanding the determinant, we get

$\therefore \Delta = (\sec\theta)(\sec\theta) - (\tan\theta)(\tan\theta)$

$= \sec^2\theta - \tan^2\theta$

$= 1$ ... $\because 1 + \tan^2\theta = \sec^2\theta \quad \therefore \sec^2\theta - \tan^2\theta = 1$

(2) Solve : (a) $\begin{vmatrix} 3 & x \\ x & 3 \end{vmatrix} = 0$ (b) $\begin{vmatrix} x^2 & -x \\ -5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & -3 \\ 5 & -3 \end{vmatrix}$

Solution :

(a) Given $\begin{vmatrix} 3 & x \\ x & 3 \end{vmatrix} = 0$

Expanding the determinant, we get

$\therefore (3)(3) - (x)(x) = 0$

$\therefore 9 - x^2 = 0$

$\therefore x^2 = 9$

$\therefore x = 3, -3.$

(b) Given $\begin{vmatrix} x^2 & -x \\ -5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & -3 \\ 5 & -3 \end{vmatrix}$

Expanding the determinants on both sides, we get

$\therefore (x^2)(1) - (-5)(-x) = (7)(-3) - (5)(-3)$

$x^2 - 5x = -21 + 15$

$x^2 - 5x + 6 = 0$ (quadratic in x)

Factorising, we get

$\therefore (x-2)(x-3) = 0$

$\therefore x = 2 \text{ or } x = 3$

.... EXERCISES ....

(1) **Evaluate the following determinants :**

(a) $\begin{vmatrix} -3 & 7 \\ -5 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} x-1 & x^2-x+1 \\ x+1 & x^2+x+1 \end{vmatrix}$ (c) $\begin{vmatrix} -x & 1 \\ -b & -a-x \end{vmatrix}$

(d) $\begin{vmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{vmatrix}$ (e) $\begin{vmatrix} \text{cosec}\,\theta & \cot\theta \\ \cot\theta & \text{cosec}\,\theta \end{vmatrix}$ (f) $\begin{vmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{vmatrix}$

(2) **Solve the following determinant equations :**

(a) $\begin{vmatrix} x & 2 \\ 2 & x \end{vmatrix} = 0$ (b) $\begin{vmatrix} x & 1 \\ 4 & x \end{vmatrix} = \begin{vmatrix} -4 & 2 \\ 2 & x \end{vmatrix}$ (c) $\begin{vmatrix} 2 & -3 \\ 4 & 3 \end{vmatrix} = \begin{vmatrix} x & 1 \\ -2 & x \end{vmatrix}$

(d) $\begin{vmatrix} x & x \\ 6 & x \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$ (e) $\begin{vmatrix} x-1 & 2 \\ 3 & 4 \end{vmatrix} = 3$ (f) $\begin{vmatrix} x & 2 \\ 8 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$ [W'07]

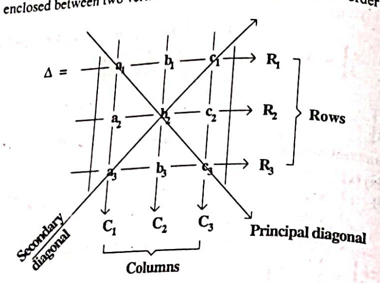
.... ANSWERS ....

(1) (a) 23 (b) −2 (c) $x^2 + ax + b$ (d) $\sin^2\theta$ (e) 1 (f) $\sec^2\theta$

(2) (a) $x = -2, 2$ (b) $x = 0, -4$ (c) $x = \pm 4$ (d) $x = 1, 5$ (e) $x = \frac{13}{4}$ (f) $x = 4$

3.3 DEFINITION OF 3 × 3 DETERMINANT

An arrangement of nine elements or numbers $a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$ in three hor[...] three vertical lines enclosed between two vertical bars is called determinant of order 3 × 3 (three b[...] simply 3. Thus,



In the above determinant,

Elements a_1, b_1, c_1 constitute the first row (R_1)
Elements a_2, b_2, c_2 constitute the second row (R_2)
Elements a_3, b_3, c_3 constitute the third row (R_3)
Elements a_1, a_2, a_3 constitute the first column (C_1)
Elements b_1, b_2, b_3 constitute the second column (C_2)
Elements c_1, c_2, c_3 constitute the third column (C_3)
Elements a_1, b_2, c_3 constitute the principal diagonal
Elements a_3, b_2, c_1 constitute the secondary diagonal.

3.3.1 Determinant of Signs

The following pattern of signs is associated with elements of 3 × 3 order determinant [...] arrangement.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

3.3.2 Minor and Co-factor of an Element in 3 × 3 Determinant

In the determinant $|A|$, *let the ith row and jth column be deleted and a new determinant is [...] having (n −1) rows and columns. This new determinant is called the minor of element* a_{ij} *and is deno[...]* M_{ij}.

Thus, if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

[...] minor of the first row, first column element i.e. minor of element a_1 is $M_{11} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

Co-factor A_{ij} of the element a_{ij} is the signed minor of a_{ij} determined by the rule,

$A_{ij} = (-1)^{i+j} \cdot M_{ij}$

[...] instance,

$A_{21} = (-1)^{2+1} \cdot M_{21} = (-1)^3 M_{21} = -M_{21}$

[...]ther words, minor of any element with sign from the determinant of signs is called a co-factor of [...]nent.

[...] following table explains how the minor and the co-factor of an element of 3 × 3 determinant is [...].

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

[...]nent	Minor of an element	Pattern of signs for co-factors	Co-factor of an element
[...]1	$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$		$+\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$
[...]1	$\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$		$-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$
[...]1	$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$		$+\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$
[...]2	$\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$	$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$	$-\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$
[...]2	$\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$		$+\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$
[...]2	$\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$		$-\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$
[...]3	$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$		$+\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$
[...]3	$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$		$-\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$
[...]3	$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$		$+\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Value of 3 × 3 Order Determinant

[...]e value of a determinant can be written down in terms of any one row or column as below.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

[...]nsidering first row, its value is

$$= a_1 \times \text{Minor of } a_1 - b_1 \times \text{Minor of } b_1 + c_1 \times \text{Minor of } c_1$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

[...]is is called the expansion of a determinant **first row wise**.

$$\therefore \Delta = a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$$

It is also known as the reduction of 3 × 3 determinant to 2 × 2 determinant. The same pro... performed with any row or column of the determinant. The pattern of signs must be beared in m...

3.3.4 3 × 3 Order Determinant Equation

Any 3 × 3 determinant with at least one unknown (variable) when equated to zero becomes a... equation.

For instance, $\begin{vmatrix} x & 2 & 1 \\ 3 & 0 & 1 \\ 4 & -5 & 2 \end{vmatrix} = 0$, where x is unknown is called a determinant equation.

The value of 'x' is obtained by expanding the determinant and equating it to zero.

.... ILLUSTRATIVE EXAMPLES ....

(1) State whether the following are determinant or not :

(a) $\begin{vmatrix} 3 & -2 & 1 \\ 0 & 4 & 5 \\ 7 & 8 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 3 & 4 \\ 7 & 0 & -2 \end{vmatrix}$ (c) $\begin{vmatrix} 2 & 3 & 0 \\ -1 & 11 & 12 \\ 5 & 7 & 6 \\ 8 & 3 & 1 \end{vmatrix}$

Solution :

(a) Determinant is a square arrangement, i.e. it has equal number of rows and colu... arrangement is a determinant since there are 3 rows and 3 columns, i.e. the number of rows and co... equal.

(b) It is not a determinant since the number of rows and columns are not equal in this arrange...

(c) It is also not a determinant as the number of rows is not equal to the number of column... arrangement.

(2) Find the value of the determinant by the expansion :

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 3 & -4 \\ 5 & -5 & 4 \end{vmatrix}$$

Solution :

$$\text{Let } \Delta = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 3 & -4 \\ 5 & -5 & 4 \end{vmatrix}$$

Expanding the determinant first row wise, we get

$$\Delta = 2\,(\text{Minor of } 2) - (-1)\,(\text{Minor of } -1) + (3)\,(\text{Minor of } 3)$$

$$= 2\begin{vmatrix} 3 & -4 \\ -5 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -4 \\ 5 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 5 & -5 \end{vmatrix}$$

$$= 2\,\{12 - (20)\} + 1\,\{4 - (-20)\} + 3\,\{-5 - 15\}$$

$$= 2\,(-8) + (24) + 3\,(-20)$$

$$= -16 + 24 - 60$$

$$= -52$$

(3) Find 'x' if

(a) $\begin{vmatrix} x & 4 & -4 \\ 3 & -2 & 1 \\ -2 & -4 & 4 \end{vmatrix} = 0$ (b) $\begin{vmatrix} 3 & 4 & 3 \\ 5 & x & x \\ 1 & 3 & 2 \end{vmatrix} = 0$

Solution :

(a) Given $\begin{vmatrix} x & 4 & -4 \\ 3 & -2 & 1 \\ -2 & -4 & 4 \end{vmatrix} = 0$

Expanding the determinant on L.H.S. first row wise, we get

$\therefore$ $x\,\{-8 + 4\} - 4\,\{12 + 2\} + (-4)\,\{-12 - 4\} = 0$

$\therefore$ $-4x - 56 + 64 = 0$

$\therefore$ $-4x + 8 = 0$

$\therefore$ $-4x = -8$

$\therefore$ $x = 2$

(b) Given $\begin{vmatrix} 3 & 4 & 3 \\ 5 & x & x \\ 1 & 3 & 2 \end{vmatrix} = 0$

Expanding the determinant on L.H.S. first row wise, we get

$\therefore$ $3\,\{2x - 3x\} - 4\,\{10 - x\} + 3\,\{15 - x\} = 0$

$\therefore$ $-3x - 40 + 4x + 45 - 3x = 0$

$\therefore$ $-2x + 5 = 0$

$\therefore$ $-2x = -5$

$\therefore$ $x = \frac{5}{2}$

(4) Solve :

(a) $\begin{vmatrix} 2 & 3 & x \\ 1 & 0 & 3 \\ -2 & -1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 8 \\ 2 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$

Solution :

(a) Given $\begin{vmatrix} 2 & 3 & x \\ 1 & 0 & 3 \\ -2 & -1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 8 \\ 2 & 1 \end{vmatrix}$

Expanding the determinants on both sides, we get

$\therefore$ $2\,\{0 + 3\} - 3\,\{0 + 6\} + x\,(-1 - 0) = -1 - 16$

$\therefore$ $6 - 18 - x = -17$

$\therefore$ $-12 - x = -17$

$\therefore$ $-x = -17 + 12$

$\therefore$ $-x = -5$

$\therefore$ $x = 5$ is the required solution.

(b) Given $\begin{vmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$

Expanding the determinants on both sides, we get

$\therefore \quad 1(18-12) - x(9-4) + x^2(3-2) = 2-2$

$\therefore \quad 6 - 5x + x^2 = 0$

i.e. $\quad x^2 - 5x + 6 = 0$ (quadratic in x)

Factorising, we get

$\therefore \quad (x-2)(x-3) = 0$

$\therefore \quad x = 2, \; x = 3$

$\therefore \quad x = 2, 3$ is the required solution.

.... EXERCISES ....

(1) Evaluate the following determinants, by the process of expansion :

(a) $\begin{vmatrix} 1 & 0 & 6 \\ 7 & 2 & 5 \\ 3 & 4 & 6 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & -1 & 0 \\ 1 & 3 & -4 \\ 5 & -3 & 4 \end{vmatrix}$ (c) $\begin{vmatrix} 2 & 5 & 7 \\ 3 & 2 & 8 \\ 6 & 4 & 1 \end{vmatrix}$

(d) $\begin{vmatrix} 3 & -5 & 1 \\ 1 & 3 & -5 \\ -5 & 1 & 3 \end{vmatrix}$ (e) $\begin{vmatrix} 22 & 19 & 16 \\ 14 & 11 & 8 \\ 11 & 8 & 5 \end{vmatrix}$ (f) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

(2) Solve or find 'x' if :

(a) $\begin{vmatrix} 2 & 3 & 1 \\ 6 & x & 2 \\ 4 & x & -2 \end{vmatrix} = 0$ (b) $\begin{vmatrix} 1 & 1 & 1 \\ 3 & x & 3 \\ 1 & x & 2 \end{vmatrix} = 0$

(c) $\begin{vmatrix} 0 & -7 & 2 \\ 11 & x & 10 \\ 4 & 8 & 1 \end{vmatrix} = 0$ (d) $\begin{vmatrix} 3 & x & 1 \\ 10 & -25 & 6 \\ 2x+3 & x-1 & 1 \end{vmatrix} = 0$

(e) $\begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix}$ (f) $\begin{vmatrix} x & x+1 & x+2 \\ 2 & 3 & -1 \\ 3 & -2 & 4 \end{vmatrix} = 0$

(g) $\begin{vmatrix} a+10 & a+2 & a+3 \\ a+4 & a+5 & a+6 \\ 2a+7 & a+8 & 0 \end{vmatrix} = 0$ (h) $\begin{vmatrix} 1 & -2 & 4 \\ 1 & x & x^2 \\ 4 & 6 & 9 \end{vmatrix} = \begin{vmatrix} 3 & 6 \\ -2 & -4 \end{vmatrix}$

(i) $\begin{vmatrix} 1 & 2x & 4x^2 \\ 1 & 4 & 16 \\ 1 & 1 & 1 \end{vmatrix} = 0$ (j) $\begin{vmatrix} x+3 & 2x+1 & 3x+2 \\ 3x-5 & 2x-3 & x-4 \\ 2x & 8x-7 & 14x-1 \end{vmatrix} = 0$

.... ANSWERS ....

(1) (a) 124 (b) 24 (c) 165 (d) -52 (e) 0

(f) $abc + 2fgh - af^2 - bg^2 - ch^2$

(2) (a) $x = 10$ (b) $x = 3$ (c) $x = -\frac{27}{8}$ (d) $x = -4, -\frac{1}{6}$ (e) $x = 1, 2$

(f) $x = -\frac{37}{14}$ (g) $a = -9$ (h) $x = -2, \frac{3}{2}$ (i) $x = \frac{1}{2}, 2$ (j) $x = \frac{1}{2}, 1.$

3.4 FUNDAMENTAL PROPERTIES OF DETERMINANTS

(1) The value of a determinant is not altered when all the rows are changed into columns or columns into rows.

i.e. if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ ... rows are changed into columns.

(2) If any two rows (or columns) of a determinant are intrechanged, the value of a determinant is changed in sign only.

i.e. if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ ... R_1 and R_2 are interchanged

For instance,

$\Delta = \begin{vmatrix} 2 & 3 & -5 \\ 1 & 0 & 2 \\ 4 & 6 & 7 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 2 \\ 2 & 3 & -5 \\ 4 & 6 & 7 \end{vmatrix}$ ... Interchanging R_1 and R_2

(3) If any two rows (or columns) of a determinant are identical, the value of a determinant is zero.

i.e. if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ in which $R_1 = R_2$

$\therefore \quad \Delta = 0$

For instance,

if $\Delta = \begin{vmatrix} 5 & 7 & 5 \\ -3 & 10 & -3 \\ 2 & 13 & 2 \end{vmatrix}$ in which $C_1 = C_3$

$= 0$

(4) If all the elements of any one row (or column) be multiplied (or divided) by any constant number, the whole determinant is multiplied (or divided) by that number.

i.e. if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and $\Delta' = \begin{vmatrix} Ka_1 & b_1 & c_1 \\ Ka_2 & b_2 & c_2 \\ Ka_3 & b_3 & c_3 \end{vmatrix}$ in which every element of C_1 of Δ is multiplied by K.

$\therefore$ By property,

$\therefore \quad \Delta' = K\Delta$ or $\Delta = \frac{1}{K}\Delta$

It follows that if elements of any row or column have any common factor, then it can be taken out of the determinant.

For instance,

(a) if $\Delta = \begin{vmatrix} 2 & 3 & -4 \\ -8 & -12 & 16 \\ 0 & 2 & 1 \end{vmatrix}$ in which –4 is a common factor from R_2

$$\Delta = -4 \begin{vmatrix} 2 & 3 & -4 \\ 2 & 3 & -4 \\ 0 & 2 & 1 \end{vmatrix}$$

(b) if $\Delta = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ x & y & z \\ p & q & r \end{vmatrix}$

$= \begin{vmatrix} 0 & a-b & (a-b)(a+b) \\ x & y & z \\ p & q & r \end{vmatrix}$ ... Note that $a^2 - b^2 = (a+b)(a+b)$

$= (a-b) \begin{vmatrix} 0 & 1 & a+b \\ x & y & z \\ p & q & r \end{vmatrix}$ ... (a – b) is common factor from R_1.

It is observed that this constant factor can be introduced in any row or column as required. Thus,

$$\Delta = K \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$= \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ ... in which K is introduced in R_1

$= \begin{vmatrix} a_1 & Kb_1 & c_1 \\ a_2 & Kb_2 & c_2 \\ a_3 & Kb_3 & c_3 \end{vmatrix}$ ... in which K is introduced in C_2.

(5) If each element of any row (or column) be expressed as a sum (or difference) of two or more terms, then the determinant can be expressed as the sum (or difference) of two or more determinants.

Thus,

if $\Delta = \begin{vmatrix} a_1 + x & b_1 & c_1 \\ a_2 + y & b_2 & c_2 \\ a_3 + z & b_3 & c_3 \end{vmatrix}$ in which elements of R_1 are sum of two terms.

Then, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & b_1 & c_1 \\ y & b_2 & c_2 \\ z & b_3 & c_3 \end{vmatrix}$

The converse is also true.

(6) If each element of any row (or column) be multiplied by the same number and this product is added to or subtracted from the corresponding elements of any other row (or column), then the value of determinant remains unchanged.

Thus,

if $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + Ka_2 & b_1 + Kb_2 & c_1 + Kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

K - times the elements of R_2 are added to the corresponding elements of R_1, the value of Δ is not changed.

For instance,

$$\Delta = \begin{vmatrix} 2 & 3 & 1 \\ 12 & 14 & 13 \\ 10 & 11 & 12 \end{vmatrix}$$

By $R_2 - R_3$,

$$\Delta = \begin{vmatrix} 2 & 3 & 1 \\ 2 & 3 & 1 \\ 10 & 11 & 12 \end{vmatrix}$$

$= 0$ ... as $R_1 = R_2$

(7) If all the elements of any one row (or column) are zeros, the value of the determinant is zero.

For instance,

if $\Delta = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 2 & -4 \\ 3 & -1 & 5 \end{vmatrix}$,

Then $\Delta = 0$ since all the elements in the first row are zeros.

(8) If any two elements of a row (or column) are zeros, the value of a determinant is equal to the product of the third element in that row or column and its Co-factor.

For instance,

if $\Delta = \begin{vmatrix} 3 & 2 & 4 \\ 0 & -1 & 3 \\ 0 & 5 & 6 \end{vmatrix}$,

then $\Delta = 3 \begin{vmatrix} -1 & 3 \\ 5 & 6 \end{vmatrix}$ since two elements of C_1 are zeros.

$= 3\{-6 - 15\}$

$= 3\{-21\} = -63.$

3.5 SOLUTION OF SIMULTANEOUS EQUATIONS USING DETERMINANTS (CRAMER'S RULE)

(A) Cramer's Rule for two equations in two variables :

Let the two equations in two variables be

$$a_1x + b_1y = c_1$$

and $$a_2x + b_2y = c_2$$

Then, they can be expressed in terms of determinants of two by two order as below :

$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, a determinant of coefficients of x and y.

$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, a determinant obtained from D on replacing a_1, a_2 by c_1, c_2.

$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$, a determinant obtained from D on replacing b_1, b_2 by c_1, c_2.

Then, $x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$, provided $D \neq 0$

This is called Cramer's Rule for solution of simultaneous equations in two unknowns.

(B) Cramer's Rule for three equations in three variables :

Above Cramer's Rule can be extended to solve the three equations in three unknowns as below :

Let the three equations in three variables be

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3.$$

Then, they can be expressed in terms of determinants of three by three order as below :

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$, a determinant of coefficients of x, y, z.

D_x = the Determinant obtained from D on replacing a_1, a_2, a_3 by d_1, d_2, d_3.

$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

D_y = the Determinant obtained from D on replacing b_1, b_2, b_3 by d_1, d_2, d_3.

$$= \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

D_z = the Determinant obtained from D on replacing c_1, c_2, c_3 by d_1, d_2, d_3.

$$= \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Then $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$, provided $D \neq 0$

Note : Before applying Cramer's Rule, the constant terms in the equations are taken on the right hand of the equations.

.... ILLUSTRATIVE EXAMPLES ....

(1) Solve the equations by using determinant :

$$3x - 2y + 3 = 0 \quad \text{and} \quad 2x + y = 5$$

Solution :

Rewriting the equations in standard form i.e. $ax + by = c$,

$$3x - 2y = -3$$
$$2x + y = 5$$

Then, writing into D, D_x, D_y as required in Cramer's Rule, we have :

$$D = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = (3)(1) - (2)(-2) = 3 + 4 = 7$$

$$D_x = \begin{vmatrix} -3 & -2 \\ 5 & 1 \end{vmatrix} = (-3)(1) - (5)(-2) = -3 + 10 = 7$$

$$D_y = \begin{vmatrix} 3 & -3 \\ 2 & 5 \end{vmatrix} = (3)(5) - (2)(-3) = 15 + 6 = 21$$

Then, by Cramer's Rule,

$$x = \frac{D_x}{D} = \frac{7}{7} = 1$$

and $$y = \frac{D_y}{D} = \frac{21}{7} = 3$$

∴ solution is : {(1, 3)}

(2) Solve using determinant :

$$\frac{3}{x-2} + \frac{2}{y-3} = 5, \quad \frac{4}{x-2} - \frac{1}{y-3} = 3$$

Solution :

The given equations are not in the linear form $ax + by = c$. They first should be converted in linear form by proper substitution. For this purpose,

Let $\frac{1}{x-2} = a$ and $\frac{1}{y-3} = b$.

Then, the equations become,

$$3a + 2b = 5$$
$$4a - b = 3$$

Now, writing in D, D_a, D_b, we have :

$$D = \begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix} = (3)(-1) - (4)(2) = -3 - 8 = -11$$

$$D_a = \begin{vmatrix} 5 & 2 \\ 3 & -1 \end{vmatrix} = (5)(-1) - (3)(2) = -5 - 6 = -11$$

$$D_b = \begin{vmatrix} 3 & 5 \\ 4 & 3 \end{vmatrix} = (3)(3) - (4)(5) = 9 - 20 = -11$$

Then, by Cramer's Rule,

$$a = \frac{D_a}{D} = \frac{-11}{-11} = 1$$

and $$b = \frac{D_b}{D} = \frac{-11}{-11} = 1$$

But $$a = \frac{1}{x-2} \quad \therefore \quad \frac{1}{x-2} = \frac{1}{1} \Rightarrow x - 2 = 1 \Rightarrow x = 3$$

and $$b = \frac{1}{y-3} \quad \therefore \quad \frac{1}{y-3} = \frac{1}{1} \Rightarrow y - 3 = 1 \Rightarrow y = 4$$

∴ solution is : {(3, 4)}

(3) Solve the following equations using Cramer's Rule.

$$x + y + z - 6 = 0, \quad 2x + y - 2z + 2 = 0, \quad x + y - 3z + 6 = 0$$

Solution :

Rewriting the equations in the standard form, we have :

$$x + y + z = 6$$
$$2x + y - 2z = -2$$
$$x + y - 3z = -6$$

Writing them in D, D_x, D_y, D_z, we have :

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 1 & 1 & -3 \end{vmatrix} = 1(-3+2) - 1(-6+2) + 1(2-1) = -1 + 4 + 1 = 4$$

$$D_x = \begin{vmatrix} 6 & 1 & 1 \\ -2 & 1 & -2 \\ -6 & 1 & -3 \end{vmatrix} = 6(-3+2) - 1(6-12) + 1(-2+6) = -6 + 6 + 4 = 4$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & -2 & -2 \\ 1 & -6 & -3 \end{vmatrix} = 1(6-12) - 6(-6+2) + 1(-12+2) = -6 + 24 - 10 = 8$$

$$D_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 1 & -2 \\ 1 & 1 & -6 \end{vmatrix} = 1(-6+2) - 1(-12+2) + 6(2-1) = -4 + 10 + 6 = 12$$

Then, by Cramer's Rule,

$$x = \frac{D_x}{D} = \frac{4}{4} = 1$$

$$y = \frac{D_y}{D} = \frac{8}{4} = 2$$

$$z = \frac{D_z}{D} = \frac{12}{4} = 3$$

∴ solution is : {(1, 2, 3)}.

(4) Solve by Cramer's Rule :

$x + y = 3, \quad y + z = 5, \quad z + x = 4$

Solution :

Rewriting the equations in standard form introducing zero as coefficient for missing unknown, we have

$$x + y + 0 \cdot z = 3$$
$$0 \cdot x + y + z = 5$$
$$x + 0 \cdot y + z = 4$$

Writing them in D, D_x, D_y, D_z, we have :

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1(1-0) - 1(0-1) = 1 + 1 = 2$$

$$D_x = \begin{vmatrix} 3 & 1 & 0 \\ 5 & 1 & 1 \\ 4 & 0 & 1 \end{vmatrix} = 3(1-0) - 1(5-4) = 3 - 1 = 2$$

$$D_y = \begin{vmatrix} 1 & 3 & 0 \\ 0 & 5 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 1(5-4) - 3(0-1) = 1 + 3 = 4$$

$$D_z = \begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 5 \\ 1 & 0 & 4 \end{vmatrix} = 1(4-0) - 1(0-5) + 3(0-1) = 4 + 5 - 3 = 6$$

Then, by Cramer's Rule,

$$x = \frac{D_x}{D} = \frac{2}{2} = 1$$

$$y = \frac{D_y}{D} = \frac{4}{2} = 2$$

$$z = \frac{D_z}{D} = \frac{6}{2} = 3$$

∴ solution is : {(1, 2, 3)}.

(5) Using Cramer's Rule find x :

$x + y + z = 1, \quad 2x + 3y + z - 4 = 0, \quad 4x + z + 9y = 16$

Solution :

Rewriting the equations in standard form, we have :

$$x + y + z = 1$$
$$2x + 3y + z = 4$$
$$4x + 9y + z = 16$$

Note : We are asked to find x only. Hence, write only D and D_x of Cramer's Rule.

Writing them in D, D_x, we have :

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 4 & 9 & 1 \end{vmatrix} = 1(3-9) - 1(2-4) + 1(18-12) = -6 + 2 + 6 = 2$$

$$D_x = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 1 \\ 16 & 9 & 1 \end{vmatrix} = 1(3-9) - 1(4-16) + 1(36-48) = -6 + 12 - 12 = -6$$

Then, by Cramer's Rule,

$$x = \frac{D_x}{D} = \frac{-6}{2} = -3$$

$x = -3$ is the required value.

(6) Solve by Cramer's Rule :

$xy + yz + zx = xyz, \quad 2xy + 3yz + zx = 4xyz, \quad 4xy + 9yz + zx = 16xyz$

Solution :

Given equations are :

$$xy + yz + zx = xyz$$
$$2xy + 3yz + zx = 4xyz$$
$$4xy + 9yz + zx = 16xyz$$

Now, dividing the equations throughout by xyz, we get

$$\therefore \quad \frac{1}{z} + \frac{1}{x} + \frac{1}{y} = 1$$

$$\therefore \quad \frac{2}{z} + \frac{3}{x} + \frac{1}{y} = 4$$

$$\therefore \quad \frac{4}{z} + \frac{9}{x} + \frac{1}{y} = 16$$

To convert them in linear form, substitute $\frac{1}{x} = p, \ \frac{1}{y} = q, \ \frac{1}{z} = r$

$$\therefore \quad p + q + r = 1$$
$$3p + q + 2r = 4$$
$$9p + q + 4r = 16$$

Now, writing into D, D_p, D_q, D_r, we get :

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 9 & 1 & 4 \end{vmatrix} \begin{aligned} &= 1(4-2) - 1(12-18) + 1(3-9) \\ &= 2 + 6 - 6 \\ &= 2 \end{aligned}$$

$$D_p = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 2 \\ 16 & 1 & 4 \end{vmatrix} \begin{aligned} &= 1(4-2) - 1(16-32) + 1(4-16) \\ &= 2 + 16 - 12 \\ &= 6 \end{aligned}$$

$$D_q = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 2 \\ 9 & 16 & 4 \end{vmatrix} \begin{aligned} &= 1(16-32) - 1(12-18) + 1(48-36) \\ &= -16 + 6 + 12 \\ &= 2 \end{aligned}$$

$$D_r = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1 & 4 \\ 9 & 1 & 16 \end{vmatrix} \begin{aligned} &= 1(16-4) - 1(48-36) + 1(3-9) \\ &= 12 - 12 - 6 \\ &= -6 \end{aligned}$$

Then, by Cramer's Rule,

$$p = \frac{D_p}{D} = \frac{6}{2} = 3$$
$$q = \frac{D_q}{D} = \frac{2}{2} = 1$$
$$r = \frac{D_r}{D} = \frac{-6}{2} = -3$$

But $p = \frac{1}{x} \quad \therefore \frac{1}{x} = \frac{3}{1} \quad \therefore x = \frac{1}{3}$ (By invertendo)

$q = \frac{1}{y} \quad \therefore \frac{1}{y} = \frac{1}{1} \quad \therefore y = 1$

$r = \frac{1}{z} \quad \therefore \frac{1}{z} = \frac{-3}{1} \quad \therefore z = \frac{-1}{3}$

$\therefore$ solution is : $\left\{\frac{1}{3}, 1, \frac{-1}{3}\right\}$.

(7) The following equations are obtained as a result of experiment :

$$P_1 + P_2 - P_3 = 0$$
$$2P_1 + P_2 + P_3 = 26$$
$$P_2 + P_3 = 14$$

find the values of P_1, P_2, P_3.

Solution :

Rewriting the equations are

$$P_1 + P_2 - P_3 = 0$$
$$2P_1 + P_2 + P_3 = 26$$
$$0 \cdot P_1 + P_2 + P_3 = 14$$

... Note the introduction of 0 as coefficient of P_1 in third equation.

Writing them in D, D_{P_1}, D_{P_2}, D_{P_3}, we have :

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \begin{aligned} &= 1(1-1) - 1(2-0) - 1(2-0) \\ &= 0 - 2 - 2 \\ &= -4 \end{aligned}$$

$$D_{P_1} = \begin{vmatrix} 0 & 1 & -1 \\ 26 & 1 & 1 \\ 14 & 1 & 1 \end{vmatrix} \begin{aligned} &= -1(26-14) - 1(26-14) \\ &= -12 - 12 \\ &= -24 \end{aligned}$$

$$D_{P_2} = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 26 & 1 \\ 0 & 14 & 1 \end{vmatrix} \begin{aligned} &= 1(26-14) - 1(28-0) \\ &= 12 - 28 \\ &= -16 \end{aligned}$$

$$D_{P_3} = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 26 \\ 0 & 1 & 14 \end{vmatrix} \begin{aligned} &= 1(14-26) - 1(28-0) \\ &= -12 - 28 \\ &= -40 \end{aligned}$$

Then, by Cramer's Rule of determinants, we have :

$$P_1 = \frac{D_{P_1}}{D} = \frac{-24}{-4} = 6$$
$$P_2 = \frac{D_{P_2}}{D} = \frac{-16}{-4} = 4$$
$$P_3 = \frac{D_{P_3}}{D} = \frac{-40}{-4} = 10$$

$\therefore$ The required values of P are : $P_1 = 6, \ P_2 = 4, \ P_3 = 10$.

(8) The following equations are obtained in electrical experiment. Find I_3 by Cramer's Rule :
$4I_1 - 3I_2 = 2, \quad 9I_2 - 4I_3 = 3I_1, \quad -4I_2 + 9I_3 = 0$

Solution :

Rewriting the equations in standard form, we get

$$4I_1 - 3I_2 + 0 \cdot I_3 = 2$$
$$3I_1 - 9I_2 + 4I_3 = 0$$
$$0I_1 - 4I_2 + 9I_3 = 0$$

Writing them in D, and D_{I_3}, we have :

$$D = \begin{vmatrix} 4 & -3 & 0 \\ 3 & -9 & 4 \\ 0 & -4 & 9 \end{vmatrix} \begin{aligned} &= 4(-81+16) + 3(27-0) \\ &= -260 + 81 \\ &= -179 \end{aligned}$$

$$D_{I_3} = \begin{vmatrix} 4 & -3 & 2 \\ 3 & -9 & 0 \\ 0 & -4 & 0 \end{vmatrix} \begin{aligned} &= \text{Expanding } C_3 \text{ -wise} \\ &= 2(-12-0) \\ &= -24 \end{aligned}$$

Then, by Cramer's Rule,

$$I_3 = \frac{D_{I_3}}{D} = \frac{-24}{-179} = \frac{24}{179} = 0.134 \text{ A}$$

.... EXERCISES ....

(1) Solve using determinants :

(a) $2x + 3y + 1 = 0$
$3x - 2y - 5 = 0$

(b) $\frac{3}{x} - \frac{4}{y} = 2, \ \frac{1}{x} + \frac{3}{y} = 5$

(2) **Solve the following equations using Cramer's Rule of determinants :**

(a) $x + y + z = 6,\quad 3x + 3y + z = 12,\quad 2x + 3y + 2z = 14$

(b) $2x + y + z = 0,\quad x + y + 2z = 0,\quad 5x + 3y + 3z = 2$

(c) $x - y + z = 0,\quad 2x - y + z = 1,\quad x + y + z = 4$

(d) $3x = y + z,\quad 2x + y - z = 4,\quad x - y + z = 3$

(e) $4x + 3y + 5z = 10,\quad 3x + 2y + z = 0,\quad 5x + 6y + 7z = 0$

(f) $yz + 2xz + xy = 2xyz,\quad 3yz - 4xz - 2xy = xyz,\quad 2yz + 5xz - 2xy = 3xyz$

(g) $3x + y + z = 4,\quad 2x - 3y + z = 7,\quad x + y + 3z = 6$

(h) $x + y + z = 3,\quad x - y + z = 1,\quad x + y - 2z = 0$

(3) **Do as directed :**

(a) Solve by Cramer's Rule to find x if
$2x + 3y = 5,\quad y - 3z = -2,\quad z + 3x = 4$

(b) Solve by Cramer's Rule of find x if
$x + z = 4,\quad y + z = 2,\quad x + y = 0$

(c) Find z using Cramer's Rule if
$x + 2y + 3z = 6,\quad 2x + 4y = 7 - z,\quad 3x + 9z = 14 - 2y$

(d) Find y using Cramer's Rule of determinants
$x + 3z = 2y + 4,\quad 2x + y = 3z + 5,\quad 2z + y = 3 + x$

(e) Find y if $x + y = 5,\quad y + z = 8,\quad z + x = 7$ by Cramer's Rule.

(4) **Solve the following by Cramer's Rule :**

(a) The voltages in an electric circuit are related by the following equations :
$V_1 + V_2 + V_3 = 9,\quad V_1 - V_2 + V_3 = 3,\quad V_1 + V_2 - V_3 = 1$. Find V_1, V_2, V_3.

(b) The currents I_1, I_2 and I_3 in three loops gave the following equations :
$2I_1 - I_2 + I_3 = 0,\quad 4I_1 - I_3 = 2,\quad 2I_2 + I_3 = 2$. Find I_1, I_2 and I_3.

(5) Evaluate $\begin{vmatrix} 22 & 19 & 16 \\ 14 & 11 & 8 \\ 11 & 8 & 5 \end{vmatrix}$ using property.

.... ANSWERS ....

(1) (a) $x = 1,\ y = -1$ (b) $x = \frac{1}{2},\ y = 1$

(2) (a) $x = 1,\ y = 2,\ z = 3$ (b) $x = -2,\ y = 6,\ z = -2$

(c) $x = 1,\ y = 2,\ z = 1$ (d) $x = \frac{7}{3},\ y = \frac{19}{6},\ z = \frac{23}{6}$

(e) $x = \frac{10}{3},\ y = \frac{-20}{3},\ z = \frac{10}{3}$ (f) $x = 1,\ y = 3,\ z = 3$

(g) $x = 1,\ y = -1,\ z = 2$ (h) $x = y = z = 1$

(3) (a) $x = 1$ (b) $x = 1$ (c) $z = 1$ (d) $y = 3$ (e) $y = 3$

(4) (a) $V_1 = 2,\ V_2 = 3,\ V_3 = 4$ (b) $I_1 = \frac{1}{2},\ I_2 = 1,\ I_3 = 0$

(5) 0

●●● PROBLEMS OF BOARD PAPERS ●●●

(1) **Find x, if** $\begin{vmatrix} x & 2 \\ 8 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$. **[W'07, Marks 2]**

Ans. Expanding the determinants, we get

$\therefore\quad 4x - 16 = 2 - 2$

$\therefore\quad 4x - 16 = 0$

$\therefore\quad 4x = 16$

$\therefore\quad x = 4.$

(2) **Using Cramer's rule, find the value of x and y, if**
$x + y - z = 0;\quad 2x + y + 3z = 9;\quad x - y + z = 2.$ **[W'07, Marks 2]**

Ans. Rewriting equations one below the other, we have :

$$x + y - z = 0$$
$$2x + y + 3z = 9$$
$$x - y + z = 2$$

Writing them in D, D_x and D_y, we have :

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 3 \\ 1 & -1 & 1 \end{vmatrix} = 1\,(1 + 3) - 1\,(2 - 3) - 1\,(-2 - 1) = 4 + 1 + 3 = 8$$

$$D_x = \begin{vmatrix} 0 & 1 & -1 \\ 9 & 1 & 3 \\ 2 & -1 & 1 \end{vmatrix} = -1\,(9 - 6) - 1\,(-9 - 2) = -3 + 11 = 8$$

$$D_y = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 9 & 3 \\ 1 & 2 & 1 \end{vmatrix} = 1\,(9 - 6) - 1\,(4 - 9) = 3 + 5 = 8$$

Then, by Cramer's rule,

$$x = \frac{D_x}{D} = \frac{8}{8} = 1,\quad y = \frac{D_y}{D} = \frac{8}{8} = 1.$$

(3) **Evaluate :** $\begin{vmatrix} 22 & 19 & 16 \\ 14 & 11 & 8 \\ 11 & 8 & 5 \end{vmatrix}$ **using properties.** **[S'08, Marks 2]**

Ans. Let
$$\Delta = \begin{vmatrix} 22 & 19 & 16 \\ 14 & 11 & 8 \\ 11 & 8 & 5 \end{vmatrix}$$

(i) By $R_1 - R_2$ and $R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} 8 & 8 & 8 \\ 3 & 3 & 3 \\ 11 & 8 & 5 \end{vmatrix}$$

(ii) 8 is a C.F. from R_1 and 3 is a C.F. from R_2.

$$\Delta = 8 \times 3 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 11 & 8 & 5 \end{vmatrix}$$

$$\therefore \quad \Delta = 24 \times 0 \qquad \text{... Since } R_1 = R_2$$

$$= 0.$$

(4) Solve the equations by Cramer's rule :

$x + y + z = 3; \quad x - y + z = 1; \quad x + y - 2z = 0.$ [S'0...

Ans. Problem-2 (h) of Exercises on Page 3.18. Practice yourself. The answer is $x = y = z = ...$

(5) Solve : $\begin{vmatrix} x & 4 & -4 \\ 3 & -2 & 1 \\ -2 & -4 & 1 \end{vmatrix} = 0.$ [S'0...

Ans. Expanding the determinant on L.H.S. we get

$$\therefore \quad x(-2+4) - 4(3+2) - 4(-12-4) = 0$$

$$\therefore \quad 2x - 20 + 64 = 0$$

$$\therefore \quad 2x = -44$$

$$\therefore \quad x = -22.$$

(6) Solve by Cramer's rule :

$x + y = 3; \quad y + z = 5; \quad z + x = 4.$ [S'09, ...

Ans. Solved Problem (4) of Page 3.14. Practice yourself. The answer is $x = 1, \; y = 2, \; z = 3.$

(7) Solve : $\begin{vmatrix} x^2 & -x \\ -5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & -3 \\ 5 & -3 \end{vmatrix}.$ [W'09, ...

Ans. Solved Problem-2 (b) on Page 3.3. Practice yourself. The answer is $x = 2$ or $x = 3.$

(8) Using Cramer's rule, solve :

$2x + 3y = 5, \quad y - 3z = -2, \quad z + 3x = 4.$ [W'09, M...

Ans. Problem-3 (a) of Exercises on Page 3.18. Practice yourself. The answer is $x = 1, \; y = 1, ...$

(9) Find x, if $\begin{vmatrix} 4 & x \\ x & 4 \end{vmatrix} = 0.$ [S'10, M...

Ans. Given $\begin{vmatrix} 4 & x \\ x & 4 \end{vmatrix} = 0$

$$\therefore \quad 16 - x^2 = 0 \qquad \text{... Expanding determinant}$$

$$\therefore \quad x^2 = 16$$

$$\therefore \quad x = +4, -4.$$

(10) Solve by Cramer's rule :

$3x + 3y - z = 11, \quad 2x - y + 2z = 9$ and $4x + 3y + 2z = 25.$ [S'10, Ma...

Ans. It is similar to solved Problem (3) of Illustrative Examples on Page 3.13. Practice yourself in similar manner. The answer is $x = 2, \; y = 3, \; z = 4.$

Chapter **4**

MATRICES

4.1 DEFINITION OF A MATRIX OF ORDER m × n

When a system of m × n quantities or numbers (may be real or complex) are arranged in m horizontal lines (called rows) and n vertical lines (called columns) enclosed between a pair of square brackets is called a matrix of order m × n. The m × n numbers are called elements or members of a matrix.

Matrices are generally denoted by capital letters of the alphabet viz. A, B, C, ... L, M, ... X, Y, Z. The elements are generally denoted by corresponding small letters viz. a, b, c, ...

The matrix of order m × n is written as :

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Or in short it is written as :

$A = [a_{ij}]_{m \times n}$ where i = row index = 1, 2, 3, ..., m

and j = column index = 1, 2, 3, ..., n

Thus, a matrix of order 3 × 3 is written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Here, the elements a_{11}, a_{22} and a_{33} are called diagonal elements and the remaining elements ... non-diagonal elements.

a_{12} represents the element in the first row and the second column, a_{23} represents the element ... second row and third column.

In general, a_{ij} represents the element of a matrix A in the i^{th} row and j^{th} column.

Now, consider a matrix of order 2×3 given as below :

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 7 \end{bmatrix}$$

Then $a_{11} = -1$, $a_{22} = 0$, $a_{23} = 7$ ... so on.

Remark : Often the following notations are used to enclose the elements of a matrix.

[ ],	( ),	‖ ‖,	{ }
↑	↑	↑	↑
Pair of rectangular brackets.	Brace brackets	Pairs of bar brackets	Pair of curly brackets

4.2 TYPES OF MATRICES

Broadly, matrices are divided into three groups :

(a) Rectangular matrices,

(b) Square matrices,

(c) Null matrices.

Let us discuss each of them in details.

4.2.1 Rectangular Matrices (m ≠ n)

Any matrix of order $m \times n$, where $m \neq n$, i.e. the number of rows is not equal to the number of columns, is called a rectangular matrix.

Examples :

(1) $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 0 & 1 \end{bmatrix}$ is a rectangular matrix of order 3×2.

(2) $B = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 7 \end{bmatrix}$ is a rectangular matrix of order 2×3.

Special Cases :

(a) A Row Matrix OR Row Vector :

When $m = 1$, the order of a matrix is $1 \times n$. Thus, any matrix having only one row and many columns is called a row matrix or a row vector.

Examples :

(1) $A = [5]$ is a row matrix of order 1×1

(2) $B = [3 \quad 5]$ is a row matrix of order 1×2

(3) $C = [-1 \quad 0 \quad 1]$ is a row matrix of order 1×3.

(b) Column Matrix OR Column Vector :

When $n = 1$, the order of a matrix is $m \times 1$. Thus, any matrix having only one column and many rows is called a column matrix or a column vector.

Examples :

(1) $A = [2]$ is a column matrix of order 1×1

(2) $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a column matrix of order 2×1

(3) $C = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is a column matrix of order 3×1.

4.2.2 Square Matrices (m = n)

Any $m \times n$ matrix, where $m = n$, i.e. the number of rows is equal to the number of columns, is called square matrix. Its order is written as $n \times n$ or simply by n. Thus, a square matrix of order 3 means it has 3 rows and 3 columns.

Examples :

(1) $A = [3]$ is a square matrix of order 1×1 or 1.

(2) $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a square matrix of order 2.

(3) $C = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -1 & 1 \\ 4 & 3 & 0 \end{bmatrix}$ is a square matrix of order 3.

Square matrices are further sub-divided as below :

(a) Diagonal matrix

(b) Scalar matrix

(c) Unit matrix or Identity matrix

(d) Triangular matrix : These are of two types :

   (1) Upper triangular matrix

   (2) Lower triangular matrix

(e) Symmetric matrix

(f) Skew - symmetric matrix

Each of this square matrix is explained in details as under :

Diagonal Matrix : Consider a square matrix A of order 3 as below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

An imaginary line through the elements a_{11}, a_{22}, a_{33} is called principal diagonal or leading diagonal. The elements a_{11}, a_{22}, a_{33} are called leading elements or diagonal elements.

A square matrix in which every non-diagonal element is equal to zero is called a diagonal matrix. The diagonal elements may or may not be equal to zero. It is generally denoted by D.

Examples :

(1) $D = [3]$ is a diagonal matrix of order 1×1

(2) $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ is a diagonal matrix of order 2 × 2

(3) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix of order 3 × 3.

Special case of diagonal matrix is a scalar matrix.

Scalar Matrix :

A diagonal matrix in which all the diagonal elements are equal is called a scalar matrix, i by K.

Examples :

(1) $K = [2]$ is a scalar matrix of order 1 × 1

(2) $K = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is a scalar matrix of order 2 × 2

(3) $K = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ is a scalar matrix of order 3 × 3.

Note : A scalar matrix is a diagonal matrix but a diagonal matrix need not be a scalar matrix.

Unit Matrix or Identity Matrix :

A square matrix in which every diagonal element is equal to one (i.e. Unity) and every element is zero is called a unit matrix or Identity matrix. It is denoted by I.

Examples :

(1) $I = [1]$ is a unit matrix of order 1 × 1

(2) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2 × 2

(3) $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3 × 3.

Triangular Matrix :

In a square matrix, if the elements below or above the leading diagonal are all zero, it is triangular matrix. They are of two types.

(1) Upper Triangular Matrix :

In a square matrix, if the elements below the leading diagonal are all zeros, it is called upper matrix. It is denoted by 'U'.

Examples :

$U = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 6 \\ 0 & 0 & -1 \end{bmatrix}$ is a upper triangular matrix of order 3 × 3

(2) Lower Triangular Matrix :

In a square matrix, if the elements above the leading diagonal are all zeros, it is called the triangular matrix. It is denoted by 'L'.

Examples :

$L = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 6 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ is a lower triangular matrix of order 3 × 3

Symmetric Matrix :

A square matrix $A = [a_{ij}]$ in which $a_{ij} = a_{ji}$, for all values of i and j, is said to be symmetric matrix.

Examples :

$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix.

Note that the elements h, g, f are reflected as mirror image about diagonal elements a, b, c. Thus, it is symmetrical about leading diagonal.

Skew-symmetric Matrix :

A square matrix $A = [a_{ij}]$ in which $a_{ij} = -a_{ji}$, for all values of i and j, and all elements in a leading diagonal are zeros, is called a skew-symmetric matrix.

Examples :

$A = \begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Observe that all diagonal elements are zeros and diagonally opposite elements are opposite in signs.

.2.3 Zero Matrices or Null Matrices

In a matrix, if each element equal to zero, it is called a null matrix or zero matrix. It is denoted by 0.

Examples :

(1) $0 = [0]$ is a zero matrix of order 1 × 1

(2) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a zero matrix of order 2 × 2

(3) $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 2 × 3

(4) $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 3 × 3

(5) $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a zero matrix of order 3 × 1.

.3 ALGEBRA OF MATRICES

.3.1 Scalar Multiplication

If A is any matrix and K is any scalar number, then K · A is called scalar multiplication of a matrix.

Thus, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and 'K' is any scalar number,

then $K \cdot A = K \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} Ka & Kb \\ Kc & Kd \end{bmatrix}$

We observe that in scalar multiplication every element of matrix A is multiplied by K.

For instance,

(1) If $A = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}$ then $3A = 3 \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 9 & 12 \end{bmatrix}$

(2) If $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & -1 \end{bmatrix}$ then $2A = 2\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 6 \\ 8 & 4 & -2 \end{bmatrix}$

(3) If $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & -1 \\ -2 & 5 & 0 \end{bmatrix}$ then $-2A = \begin{bmatrix} -2 & 0 & -4 \\ -6 & -8 & 2 \\ 4 & -10 & 0 \end{bmatrix}$

Conversely, if every element of a matrix has a non-zero factor in common, it is taken ... matrix.

For instance,

if $A = \begin{bmatrix} 3 & -6 \\ 12 & 9 \end{bmatrix}$, then it can also be written as $A = 3\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$

Properties of Scalar Multiplication :

If A and B are any two matrices of the same order, 0 is a zero matrix of the same order, m ... scalars, then, we have :

(1) $m(A + B) = mA + mB$

(2) $(m + n)A = mA + nA$

(3) $0 \cdot A = 0$

(4) $m \times 0 = 0$

(5) $m(nA) = n(mA) = (mn)A$

These laws can be verified in examples.

4.3.2 Negative of a Matrix

If A is a given matrix, then the matrix obtained by multiplying with a scalar K = –1 is ... and the new matrix will have elements which are negative to the corresponding element of the ...

Thus, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the matrix $(-1)A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$

which is the negative of the matrix A.

4.3.3 Equality of Matrix

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal (i.e. A = B), if

(1) They are of the same order,

and (2) The elements in the corresponding positions of two matrices are equal i.e. $[a_{ij}] = [b_{ij}]$

For instance,

if $A = \begin{bmatrix} 3 & 2 \\ -4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ -4 & 5 \end{bmatrix}$ then A = B.

It follows that :

(1) If $A = \begin{bmatrix} x & 2 \\ 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 3 & 6 \end{bmatrix}$ and A = B, then x = –3

(2) If $\begin{bmatrix} 3 & a & 2 \\ 1 & 0 & b \end{bmatrix} = \begin{bmatrix} c & -3 & 2 \\ d & e & 5 \end{bmatrix}$, then a = –3, b = 5, c = 3, d = 1, e = 0

(3) (a) If A = B, then B = A (Symmetry)

(b) A = A, where A is any matrix (Reflectivity)

(c) If A = B, B = C, then A = C (Transitivity)

(4) With this definition of equality of matrices, we can express the equations

$a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$ } more compactly as $\begin{bmatrix} a_1x + b_1y \\ a_2x + b_2y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

in matrix form.

For instance,

$4x + 3y = 7$

$4x - 2y = 2$

Then, writing in matrix form we have

$$\begin{bmatrix} 4x + 3y \\ 4x - 2y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

Similarly, the set of equations

$x + y + z = 6$

$3x - y + 3z = 10$

$5x + 5y - 4z = 3$ could be expressed in matrix form as :

$$\begin{bmatrix} x + y + z \\ 3x - y + 3z \\ 5x + 5y - 4z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 3 \end{bmatrix}$$

4.3.4 Addition of Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ could be added to each other, if they are of the same order. Otherwise their addition is not possible.

When the matrices are of the some order, they are confirmable for addition. If the matrices are not of the same order, they are not confirmable for addition.

For instance,

If A is 2 × 3, B is 2 × 3, then A + B is defined and is a matrix of the same order 2 × 3, obtained by adding the corresponding elements of A and B.

Thus,

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2\times 3} + \begin{bmatrix} x & y & z \\ p & q & r \end{bmatrix}_{2\times 3} = \begin{bmatrix} a+x & b+y & c+z \\ d+p & e+q & f+r \end{bmatrix}_{2\times 3}$$

Similarly, if $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$,

then
$$2A + 3B = 2\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} + 3\begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & 6 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 3 & -6 \end{bmatrix}$$
$$= \begin{bmatrix} 4+(-3) & 6+0 \\ 8+3 & 2+(-6) \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 11 & -4 \end{bmatrix}$$

Properties of Matrix Addition :

A, B and C are three matrices of the same order and 0 is a zero matrix of the same order, then

(1) The addition of matrices is commutative :

i.e. A + B = B + A

(2) The addition of matrices is associative :

i.e. A + (B + C) = (A + B) + C

(3) Existence of additive identity

i.e. $A + 0 = 0 + A = A$

(4) Existence of additive inverse

i.e. $A + (-A) = 0 = (-A) + A$, then $-A$ is known as additive inverse of A.

(5) If $A + B = A + C$, then $B = C$ (Left cancellation)

(6) If $B + A = C + A$, then $B = C$ (Right cancellation)

4.3.5 Subtraction of Matrices

If two matrices A and B are of the same order, then their subtraction is defined as A – B = A + (–B). The matrix (–1) B is usually written as –B.

Thus, if A is m × n, B is m × n, then A – B is a matrix of order m × n obtained by subtracting elements of B from the corresponding elements of A.

For instance,

(1) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} - \begin{bmatrix} p & q \\ r & s \end{bmatrix}_{2\times 2} = \begin{bmatrix} a-p & b-q \\ c-r & d-s \end{bmatrix}_{2\times 2}$$

Similarly, if $A = \begin{bmatrix} 3 & 4 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$,

then
$$\begin{aligned} 3A - 2B &= 3\begin{bmatrix} 3 & 4 \\ -1 & 1 \end{bmatrix} - 2\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 12 \\ -3 & 3 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 9-(-2) & 12-4 \\ -3-0 & 3-(-2) \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ -3 & 5 \end{bmatrix} \end{aligned}$$

.... ILLUSTRATIVE EXAMPLES ....

(1) If $A = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 & -1 \\ 1 & 4 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 1 & 3 \end{bmatrix}$, compute for $3A - 2B + 2C$.

Solution : Since the matrices A, B, C are of the same order, their addition is possible.

$$\begin{aligned} \therefore \quad 3A - 2B + 2C &= 3\begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 5 \end{bmatrix} - 2\begin{bmatrix} 3 & 5 & -1 \\ 1 & 4 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & -1 & 4 \\ 2 & 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 12 & 3 \\ 0 & 9 & 15 \end{bmatrix} - \begin{bmatrix} 6 & 10 & -2 \\ 2 & 8 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 4 & 2 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 6-6+2 & 12-10-2 & 3+2+8 \\ 0-2+4 & 9-8+2 & 15-0+6 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 13 \\ 2 & 3 & 21 \end{bmatrix}_{2\times 3} \end{aligned}$$

(2) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix}$, **evaluate : 2A – 3B.**

Solution :

Matrices A and B are of the same order. Hence, their addition is confirmed.

$$\begin{aligned} \therefore \quad 2A - 3B &= 2\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 7 & 8 & 9 \end{bmatrix} - 3\begin{bmatrix} 2 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 & 6 \\ 0 & 8 & 10 \\ 14 & 16 & 18 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 9 \\ 12 & 0 & -3 \\ 6 & 9 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2-6 & 4-0 & 6-9 \\ 0-12 & 8-0 & 10+3 \\ 14-6 & 16-9 & 18-0 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 4 & -3 \\ -12 & 8 & 13 \\ 8 & 7 & 18 \end{bmatrix}_{3\times 3} \end{aligned}$$

(3) If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$, **find 2A + 3B – 4I, where I is the unit matrix of order two.**

Solution :

A is 2 × 2, B is 2 × 2 ∴ for 2A + 3B – 4I to hold, I must be 2 × 2.

$$\begin{aligned} \therefore \quad 2A + 3B - 4I &= 2\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} + 3\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 6 \\ 8 & 14 \end{bmatrix} + \begin{bmatrix} 3 & 9 \\ 12 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 4+3-4 & 6+9-0 \\ 8+12-0 & 14+18-4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 15 \\ 20 & 28 \end{bmatrix}_{2\times 2} \end{aligned}$$

(4) If $A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$, **find matrix B such that A + B = 0.**

Solution :

A is 2 × 2 ∴ A + B = 0 to be hold true, B and 0 are 2 × 2.

$$\begin{aligned} \because \quad A + B &= 0 \\ \therefore \quad B &= 0 - A = -A = -\begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \\ \therefore \quad B &= \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}_{2\times 2} \end{aligned}$$

(5) If $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -3 \end{bmatrix}$, **find matrix C such that A + B + C = 0.**

Solution :

Here A is 2 × 3, B is 2 × 3, and therefore C and 0 must be 2 × 3 to hold the statement A + B + C = 0 true.

$$\begin{aligned} \because \quad A + B + C &= 0 \\ \therefore \quad C &= 0 - (A+B) = -(A+B) \\ C &= -\left\{\begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -3 \end{bmatrix}\right\} \\ &= -\begin{bmatrix} 1+1 & -2+0 & 3+1 \\ 4-2 & 5+1 & -1-3 \end{bmatrix} \\ &= -\begin{bmatrix} 2 & -2 & 4 \\ 2 & 6 & -4 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -4 \\ -2 & -6 & 4 \end{bmatrix} \end{aligned}$$

(6) If $A = \begin{bmatrix} -2 & 4 \\ 1 & 3 \end{bmatrix}$, find matrix 'X' such that $A - 2X = \begin{bmatrix} -2 & -8 \\ 3 & -5 \end{bmatrix}$.

Solution :

A is 2 × 2 ∴ X is also 2 × 2.

Given $A - 2X = \begin{bmatrix} -2 & -8 \\ 3 & -5 \end{bmatrix}$, rewriting, we get

$$\therefore \quad 2X = A - \begin{bmatrix} -2 & -8 \\ 3 & -5 \end{bmatrix}$$

$$\therefore \quad 2X = \begin{bmatrix} -2 & 4 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} -2 & -8 \\ 3 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -2+2 & 4+8 \\ 1-3 & 3+5 \end{bmatrix}$$

$$\therefore \quad X = \frac{1}{2}\begin{bmatrix} 0 & 12 \\ -2 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -1 & 4 \end{bmatrix}$$

(7) Find 'X' Such that $3X + \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ -8 & 9 \end{bmatrix}$

Solution :

Given $3X + \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ -8 & 9 \end{bmatrix}$

Rewriting, we get

$$\therefore \quad 3X = \begin{bmatrix} 7 & 11 \\ -8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$$

$$\therefore \quad 3X = \begin{bmatrix} 7-4 & 11-5 \\ -8-1 & 9+3 \end{bmatrix}$$

$$\therefore \quad 3X = \begin{bmatrix} 3 & 6 \\ -9 & 12 \end{bmatrix}$$

$$\therefore \quad X = \frac{1}{3}\begin{bmatrix} 3 & 6 \\ -9 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

(8) If $A \begin{bmatrix} 2 & 3 & 4 \\ -3 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -4 & -5 \\ 1 & 2 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & -1 & 2 \\ 7 & 0 & 3 \end{bmatrix}$, find matrix 'X' that $2A + 3B - X = C$.

Solution :

As matrices A, B, C are all 2 × 3, X must be 2 × 3 to hold 2A + 3B − X = C.

$\because$ 2A + 3B − X = C

$$\therefore \quad X = 2A + 3B - C \quad \text{... Note the transposition of X.}$$

$$= 2\begin{bmatrix} 2 & 3 & 4 \\ -3 & 0 & 2 \end{bmatrix} + 3\begin{bmatrix} 3 & -4 & -5 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & -1 & 2 \\ 7 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & 8 \\ -6 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 9 & -12 & -15 \\ 3 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 5 & -1 & 2 \\ 7 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+9-5 & 6-12+1 & 8-15-2 \\ -6+3-7 & 0+6-0 & 4+3-3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -5 & -9 \\ -10 & 6 & 4 \end{bmatrix}_{2 \times 3}$$

(9) Find C, if A + 2C = B where $A = \begin{bmatrix} 1 & -2 & 4 \\ 5 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 3 & -1 \end{bmatrix}$

Solution :

Given A + 2C = B, rewriting, we have :

2C = B − A

$$\therefore \quad 2C = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 3 & -1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 4 \\ 5 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix}$$

$$\therefore \quad 2C = \begin{bmatrix} 2-1 & -1+2 & 3-4 \\ 4-5 & 2-0 & 5-1 \\ 0-3 & 3-2 & -1+1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 4 \\ -3 & 1 & 0 \end{bmatrix}$$

$$\therefore \quad C = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 2 & 4 \\ -3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & 1 & 2 \\ \frac{-3}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

(10) If $X = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$, $Z = \begin{bmatrix} 7 & 11 \\ -8 & 9 \end{bmatrix}$, show that 3X + Y = Z.

Solution :

Let us first find 3X + Y.

$$\therefore \quad 3X + Y = 3\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 \\ -9 & 12 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+4 & 6+5 \\ -9+1 & 12-3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 11 \\ -8 & 9 \end{bmatrix}$$

$$= Z \quad \text{... By inspection}$$

$\therefore$ 3X + Y = Z is true.

(11) If $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -1 \\ 3 & 2 \\ 4 & -2 \end{bmatrix}$, verify that A + B = B + A.

Solution :

A is 3 × 2, B is 3 × 2

$\therefore$ A + B and B + A are defined.

$$A + B = \begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 3 & 2 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-1 & 2-1 \\ 1+3 & -1+2 \\ 0+4 & 4-2 \end{bmatrix}$$

$$\therefore \quad A + B = \begin{bmatrix} 2 & 1 \\ 4 & 1 \\ 4 & 2 \end{bmatrix} \quad \ldots (1)$$

and

$$B + A = \begin{bmatrix} -1 & -1 \\ 3 & 2 \\ 4 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1+3 & -1+2 \\ 3+1 & 2-1 \\ 4+0 & -2+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 4 & 1 \\ 4 & 2 \end{bmatrix} \quad \ldots (2)$$

From results (1) and (2), we have

$$A + B = B + A \quad \text{(verified)}$$

(12) If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$,

verify that $(A + B) + C = A + (B + C)$.

Solution :

Since matrices A, B, C are all square matrices of the same order, their additions are defined.

$$\text{Consider } A + B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\therefore (A + B) + C = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} \quad \ldots (1)$$

$$\text{Next,} \quad B + C = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$$

$$\therefore A + (B + C) = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} \quad \ldots (2)$$

From results (1) and (2), we have

$$(A + B) + C = A + (B + C) \quad \text{(verified)}$$

(13) Find x and y satisfying the equation

$$\begin{bmatrix} 1 & x & 0 \\ y & 2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 5 & 2 \end{bmatrix}.$$

Solution :

Given

$$\begin{bmatrix} 1 & x & 0 \\ y & 2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 5 & 2 \end{bmatrix}.$$

Adding matrices on L.H.S., we get

$$\therefore \begin{bmatrix} 1+3 & x+1 & 0+2 \\ y+4 & 2+3 & 4-2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 5 & 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 4 & x+1 & 2 \\ y+4 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 5 & 2 \end{bmatrix}$$

Now, by equality of matrices, we get

$\therefore$ $x + 1 = 2$, $y + 4 = 6$

$\therefore$ $x = 2 - 1$, $y = 6 - 4$

$\therefore$ $x = 1$, $y = 2$

(14) If $A = \begin{bmatrix} x & 3 & -7 \\ 4 & 0 & 6y \end{bmatrix}$ and $B = \begin{bmatrix} 1-y & 9 & -21 \\ 12 & 0 & 36 \end{bmatrix}$ and if $3A = B$, find x and y.

Solution :

Given $3A = B$

$$\therefore \quad 3\begin{bmatrix} x & 3 & -7 \\ 4 & 0 & 6y \end{bmatrix} = \begin{bmatrix} 1-y & 9 & -21 \\ 12 & 0 & 36 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 3x & 9 & -21 \\ 12 & 0 & 18y \end{bmatrix} = \begin{bmatrix} 1-y & 9 & -21 \\ 12 & 0 & 36 \end{bmatrix}$$

By equality of matrices, we get

$3x = 1 - y$ and $18y = 36$ $\therefore$ $y = 2$

Putting $y = 2$ in $3x = 1 - y$ $\therefore$ $3x = 1 - 2$ $\therefore$ $3x = -1$ $\therefore$ $x = \frac{-1}{3}$

Thus, $x = \frac{-1}{3}$, $y = 2$

(15) If $A = \begin{bmatrix} -1 & 3 \\ 4 & 5 \end{bmatrix}$, express it as the sum of the symmetric and the skew-symmetric matrix.

Solution :

Let the matrix $A = B + C$ ... (1)

where B = symmetric matrix = $\begin{bmatrix} a & c \\ c & b \end{bmatrix}$, say

and C = skew-symmetric matrix = $\begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}$, say

From (1), substituting B and C, we get

$$\begin{bmatrix} -1 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} a & c \\ c & b \end{bmatrix} + \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}$$

Adding matrices on R.H.S. we get

$$\therefore \begin{bmatrix} -1 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} a & c-x \\ c+x & b \end{bmatrix}$$

Now, by equality of matrices, we get

$a = -1$, $b = 5$, $c - x = 3$, $c + x = 4$

Solving $c - x = 3$

and $c + x = 4$ ... simultaneously,

$2c = 7$

$\therefore$ $c = \frac{7}{2}$

Putting $c = \frac{7}{2}$ in $c + x = 4$ $\therefore x = 4 - c = 4 - \frac{7}{2} = \frac{1}{2}$ $\therefore x = \frac{1}{2}$

Thus, $B = \begin{bmatrix} a & c \\ c & b \end{bmatrix} = \begin{bmatrix} -1 & \frac{7}{2} \\ \frac{7}{2} & 5 \end{bmatrix}$

$C = \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} \\ +\frac{1}{2} & 0 \end{bmatrix}$

$\therefore$ $A = B + C$ gives

$$\begin{bmatrix} -1 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -1 & \frac{7}{2} \\ \frac{7}{2} & 5 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} \\ +\frac{1}{2} & 0 \end{bmatrix}$$

Symmetric matrix — Skew-symmetric matrix

.... EXERCISES ....

(1) (a) Is the sum $\begin{bmatrix} 1 & -2 & 0 \\ 4 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$ defined? Justify your answer.

(b) Evaluate : $\begin{bmatrix} 2 & -1 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ -4 & 5 \end{bmatrix}$.

(c) If $A = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 3 & -2 \end{bmatrix}$, obtain the matrix $A - 3B$.

(d) If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$, find $3A - 2B$.

(e) If $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 1 \\ 4 & -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 6 & 7 \\ 8 & 0 & 4 \\ 2 & -3 & 1 \end{bmatrix}$, find $3A + 2B$.

(f) If $A = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 7 \\ 5 & 2 \end{bmatrix}$, find $5A - 3B + 2C$.

(g) If $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$ evaluate $3A - 4B$.

(h) If $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$, find $2A + 3B - 4I$

where I is the unit matrix of order two.

(i) If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, $B = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ find $\cos\theta \cdot A + \sin\theta \cdot B$.

(2) (a) If $A = \begin{bmatrix} 2 & -3 \\ 7 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 9 \\ -5 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 6 \\ 2 & 9 \end{bmatrix}$, show that $A + B = C$.

(b) Given the matrices

$A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 4 & -5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ -2 & 5 \end{bmatrix}$

verify that $(A + B) + C = A + (B + C)$.

(c) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$, show that $A + B = 0$.

(d) If $A = \begin{bmatrix} 1 & 3 \\ 7 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 7 \\ 9 & 0 \end{bmatrix}$, show that $A + B = B + A$.

(e) If $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 5 \\ -8 & 2 \end{bmatrix}$ show that $A - 3B = C$.

(3) (a) Can we find x and y in the following matrix equation?

$\begin{bmatrix} 3x^2 & 4 \\ 1 & y - 3 \end{bmatrix} = \begin{bmatrix} 12 & 4 \\ 1 & y - 3 \end{bmatrix}$.

(b) If $2\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ -8 \end{bmatrix}$, find x and y.

(c) Find the values of x and y satisfying the following equation :

$\begin{bmatrix} 1 & x & 0 \\ y & 2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 5 & 2 \end{bmatrix}$.

(d) Find the values of x and y satisfying the matrix equation :

$\begin{bmatrix} 2x + 1 & -1 & 1 \\ 3 & 4y & 4 \end{bmatrix} + \begin{bmatrix} -1 & 6 & 4 \\ 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 5 \\ 6 & 12 & 7 \end{bmatrix}$.

(e) If $\begin{bmatrix} 2 & -3 \\ 7 & 8 \end{bmatrix} + \begin{bmatrix} a & 9 \\ -5 & b \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 2 & 9 \end{bmatrix}$, show that $a + b = 3$.

(f) Given $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 2 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$

find the value of K if $A + 2B + KC = \begin{bmatrix} 0 & 0 \\ -1 & 13 \end{bmatrix}$.

(4) (a) If $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ and $2A + 3B = 0$, then find the matrix B.

(b) If $A = \begin{bmatrix} 4 & 2 & -5 \\ -3 & 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 5 & -1 \\ 1 & 2 & 5 \end{bmatrix}$, find the matrix 'X' such that $A + X = B$.

(c) Find X if $\begin{bmatrix} 4 & 5 \\ -3 & 6 \end{bmatrix} + X = \begin{bmatrix} 10 & -1 \\ 0 & -5 \end{bmatrix}$.

(d) If $A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}$, find the matrix 'X' such that $2X + 3A - 4B = I$, where I is identity matrix of order 2.

(e) If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$, find 'X' such that $2X + 3A - 2B = 0$

(f) If $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$, find the matrix 'X' such that $2A + X = 3B$

(g) Find 'X' such that $2\left\{X + \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 0 \end{bmatrix}\right\} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

(h) Find 'X' and 'Y', if $X + Y = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

(i) Find the matrices A and B, if $A + B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$ and $A - B = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

.... ANSWERS ....

(1) (a) No, since the order of two matrices are different.

(b) $\begin{bmatrix} 1 & 2 \\ 0 & 12 \end{bmatrix}$ (c) $\begin{bmatrix} 6 & 6 \\ -7 & 7 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 3 \\ 4 & 9 \end{bmatrix}$

(e) $\begin{bmatrix} -1 & 9 & 20 \\ 16 & 3 & 11 \\ 16 & -9 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 42 \\ 4 & 9 \end{bmatrix}$ (g) $\begin{bmatrix} 2 & 1 & 27 \\ 0 & 1 & 3 \end{bmatrix}$

(h) $\begin{bmatrix} 3 & 15 \\ 20 & 28 \end{bmatrix}$. (i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(3) (a) $x = \pm 2$, y can not be determined. (b) $x = 5, \ y = -4$

(c) $x = 1, \ y = 2$ (d) $x = 2, \ y = 3$ (f) $k = -1$

(4) (a) $\frac{1}{3}\begin{bmatrix} -2 & -4 \\ 6 & -8 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 3 & 4 \\ 4 & -3 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 6 & -6 \\ 3 & -11 \end{bmatrix}$

(d) $\frac{1}{2}\begin{bmatrix} -4 & 11 \\ -18 & -11 \end{bmatrix}$ (e) $\begin{bmatrix} -4 & -\frac{5}{2} \\ \frac{3}{2} & -5 \end{bmatrix}$ (f) $\begin{bmatrix} 5 & -4 \\ -11 & 6 \end{bmatrix}$

(g) $\frac{1}{2}\begin{bmatrix} -5 & 2 & -5 \\ -7 & -5 & 1 \end{bmatrix}$ (h) $X = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$, $Y = \begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}$

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -2 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

4.3.6 The Multiplication of Matrices

(a) Inner Product :

The single equation $ax + by + cz = d$ may be written in the form

$$[a \quad b \quad c]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [d]$$

In this case, $[a \quad b \quad c]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [ax + by + cz]$

is called the inner product of the matrices $[a \quad b \quad c]$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. The linear form $ax + by + cz$ is the sum of the product of corresponding terms of the two matrices.

(b) Procedure of Multiplication of Matrices :

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be confirmable for the product AB, if the number of columns of A equals the number of rows of B.

If A and B are not confirmable for the product AB, then AB is not defined.

Symbolically, if A is $m \times p$, B is $p \times n$, then AB is possible and its order is $m \times n$.

Similarly,

if A is 2×3, B is 3×2, then AB is 2×2

if A is 3×1, B is 1×3, then AB is 3×3

if A is 3×2, B is 3×2, then AB is not defined.
not equal

Examples :

(1) $A = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix}$, find AB.

A is 2×2, B is 2×2 $\therefore$ AB is 2×2

$$\therefore AB = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}\begin{matrix} \to R_1 \\ \to R_2 \end{matrix} \cdot \begin{bmatrix} -1 & 2 \\ 0 & 4 \end{bmatrix}$$
$$\qquad\qquad\qquad\qquad\quad \downarrow C_1 \ \downarrow C_2$$

$$= \begin{bmatrix} R_1 C_1 & R_1 C_2 \\ R_2 C_1 & R_2 C_2 \end{bmatrix}$$

$$= \begin{bmatrix} (3)(-1) + (2)(0) & (3)(2) + (2)(4) \\ (4)(-1) + (-1)(0) & (4)(2) + (-1)(4) \end{bmatrix}$$

$$= \begin{bmatrix} -3 + 0 & 6 + 8 \\ -4 + 0 & 8 - 4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 14 \\ -4 & 4 \end{bmatrix}_{2 \times 2}$$

(2) $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 5 & -2 \end{bmatrix}$, find AB, if possible. 3x2 2x3

In this case, A is 3×2, B is 2×3 $\therefore$ AB is defined and is 3×3
equals

$$\therefore AB = \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 1 & -2 \end{bmatrix}\begin{matrix} \to R_1 \\ \to R_2 \\ \to R_3 \end{matrix} \cdot \begin{bmatrix} 1 & -1 & 3 \\ 4 & 5 & -2 \end{bmatrix}$$
$$\qquad\qquad\qquad\qquad\quad \downarrow C_1 \ \downarrow C_2 \ \downarrow C_3$$

$$= \begin{bmatrix} R_1 C_1 & R_1 C_2 & R_1 C_3 \\ R_2 C_1 & R_2 C_2 & R_2 C_3 \\ R_3 C_1 & R_3 C_2 & R_3 C_3 \end{bmatrix}$$

$$= \begin{bmatrix} (3)(1)+(-1)(4) & (3)(-1)+(-1)(5) & (3)(3)+(-1)(-2) \\ (2)(1)+(0)(4) & (2)(-1)+(0)(5) & (2)(3)+(0)(-2) \\ (1)(1)+(-2)(4) & (1)(-1)+(-2)(5) & (1)(3)+(-2)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 3-4 & -3-5 & 9+2 \\ 2+0 & -2+0 & 6+0 \\ 1-8 & -1-10 & 3+4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -8 & 11 \\ 2 & -2 & 6 \\ -7 & -11 & 7 \end{bmatrix}$$

(c) Divisors of Zero :

If the product of two or more matrices is zero, it does not follow that one of the factors is factors are called divisors of zero.

For instance,

If $A = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 \\ -4 & -12 \end{bmatrix}$,

then $AB = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 6 \\ -4 & -12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Here, $A \neq 0$, $B \neq 0$, but $AB = 0$. Such matrices A and B are called divisors of zero.

Properties of Matrix Multiplication :

(1) Matrix multiplication, in general is non-commutative :

If A and B are two matrices such that the products AB and BA, both are defined, then $AB \neq BA$.

(2) Matrix multiplication is always an associative :

If A, B and C are three matrices such that the respective products are defined, then A (BC) = always.

(3) Matrix multiplication is distributive over addition/subtraction :

If A and B are of the same order $m \times n$ and C is $n \times q$, then $(A + B) C = AC + BC$ always. So if E is of order $p \times m$, then $E (A + B) = EA + EB$, always.

(4) Cancellation law does not hold :

If $AB = AC$, then it does not imply $B = C$.

(5) For two diagonal matrices A and B, it is always true that $AB = BA$.

Remarks :

(1) AB and BA are both defined, if and only if A is $m \times n$ and B is $n \times m$. Thus, for square ma and B, both the product AB and BA are defined.

(2) In the product AB, B is pre-multiplied by A and A is post-multiplied by B.

(3) If A and B are $m \times n$ matrices $(m = n)$ such that $AB = BA$, then A and B are said to be com

(4) $A \cdot A$, which we write as A^2, exists only if $m = n$ i.e. A is a square matrix, A^2 is also matrix of the order same as A.

(5) If A, B and I are square matrices of the same order, then

(i) $(A + B)^2 = A^2 + AB + BA + B^2$

(ii) $(A - B)^2 = A^2 - AB - BA + B^2$

(iii) $(A + B)(A - B) = A^2 - AB + BA - B^2$

(iv) $A \cdot I = I \cdot A = A$

(6) $I = I^2 = I^3 = \ldots = I^m$, where m is any positive integer.

In fact, multiplying any matrix by a unit matrix leaves the matrix unaffected provided that the product is defined. For instance,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3} \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}_{3 \times 1}$$

(7) For a matrix A, $A \cdot 0 = 0 \cdot A = 0$ if the products are defined.

(8) If the product AB is defined and $AB = I$, then A and B are called inverse of each other.

.... ILLUSTRATIVE EXAMPLES ....

(1) If $A = \begin{bmatrix} 1 & -5 \\ 6 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, find the matrix AB − 2I.

Solution :

A and B are square matrices of order 2×2. So AB is defined and $AB - 2I$ to hold, I must be 2×2.

$$\therefore \quad AB - 2I = \begin{bmatrix} 1 & -5 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+5 \\ 6+0 & 0-4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 \\ 6 & -4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 5 \\ 6 & -6 \end{bmatrix}$$

(2) If $A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$, show that A^2 is a null matrix.

Solution :

Here $A^2 = A \cdot A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 4-4 & 8-8 \\ -2+2 & -4+4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \text{Null matrix}$$

(3) If $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$, find AB. What conclusion can be drawn from this result?

Solution :

A and B are square matrices of order 2×2. ∴ AB is defined.

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4+4 & 4-4 \\ 4-4 & -4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{Null matrix}$$

∴ AB is a null matrix.

Here $A \neq 0$, $B \neq 0$, but $AB = 0$.

Thus, in matrix algebra, if the product $AB = 0$, it does not necessarily imply that either $A = 0$ or $B = 0$.

(4) If $A = \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -6 & 4 \\ 2 & 0 & 3 \end{bmatrix}$, find the matrix AB and without comp... show that AB ≠ BA.

Solution :

In this problem,

A is 3×2, B is 2×3 (Equals; Gives the order of product) ∴ AB is defined and is 3×3.

$$\therefore \quad AB = \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 3 & -4 \end{bmatrix} \begin{matrix} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{matrix} \cdot \begin{bmatrix} 1 & -6 & 4 \\ 2 & 0 & 3 \end{bmatrix}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ C_1 & C_2 & C_3 \end{matrix}$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2 & -24+0 & 16+3 \\ 5+4 & -30+0 & 20+6 \\ 3-8 & -18+0 & 12-12 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -24 & 19 \\ 9 & -30 & 26 \\ -5 & -18 & 0 \end{bmatrix}_{3 \times 3}$$

Next, B is 2×3, A is 3×2 (Equals; Gives the order of product) ∴ BA is defined and is 2×2.

Thus, $(AB)_{3 \times 3} \neq (BA)_{2 \times 2}$.

(5) If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$, show that AB = BA = I

Solution :

A and B are square matrices of the same order. ∴ AB and BA, both are defined.

$$AB = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \ldots (1)$$

and

$$BA = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6-5 & 15-15 \\ -2+2 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \ldots (2)$$

From results (1) and (2), it is found that AB = BA = I.

(6) If $A = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}$, show that $A^2 - 8A$ is a scalar matrix.

Solution :

Here

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+16+16 & 8+8+16 & 8+16+8 \\ 8+8+16 & 16+4+16 & 16+8+8 \\ 8+16+8 & 16+8+8 & 16+16+4 \end{bmatrix}$$

$$= \begin{bmatrix} 36 & 32 & 32 \\ 32 & 36 & 32 \\ 32 & 32 & 36 \end{bmatrix}_{3 \times 3}$$

$$\therefore \quad A^2 - 8A = \begin{bmatrix} 36 & 32 & 32 \\ 32 & 36 & 32 \\ 32 & 32 & 36 \end{bmatrix} - 8 \begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 36 & 32 & 32 \\ 32 & 36 & 32 \\ 32 & 32 & 36 \end{bmatrix} - \begin{bmatrix} 16 & 32 & 32 \\ 32 & 16 & 32 \\ 32 & 32 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} \text{ which is a scalar matrix}$$

∴ $A^2 - 8A$ is a scalar matrix

(7) If $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$, show that A satisfies the matrix equation $A^2 = 3A + 2I$, where I is the identity matrix. **[W'07]**

Solution :

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+4 & 8+4 \\ 2+1 & 4+1 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 3 & 5 \end{bmatrix} \quad \ldots (1)$$

and

$$3A + 2I = 3 \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 12 \\ 3 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 3 & 5 \end{bmatrix} \quad \ldots (2)$$

From results (1) and (2), it is clear that

$A^2 = 3A + 2I$ ... (Satisfied)

(8) If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find $A^2 - 9A + 14I$, where I is a unit matrix. **[Similar to W'07]**

Solution :

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix}$$

$$\therefore \quad A^2 - 5A + 14I$$

$$= \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} - 9 \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} + 14 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} - \begin{bmatrix} 36 & 27 \\ 18 & 45 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 22 - 36 + 14 & 27 - 27 + 0 \\ 18 - 18 + 0 & 31 - 45 + 14 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \text{Null matrix}$$

(9) If $A = \begin{bmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{bmatrix}$, $B = \begin{bmatrix} x & x^2 & 1 \\ x^2 & 1 & x \\ 1 & x & x^2 \end{bmatrix}$, **where x is the complex cube root of unity, …**

AB and show that AB is a null matrix.

Solution :

x is the complex cube root of unity.

Let $x = \sqrt[3]{1} \quad \therefore x^3 = 1 \quad \therefore x^3 - 1 = 0$

Factorizing,

$\therefore \quad (x-1)(x^2 + x + 1) = 0$

$\therefore \quad x - 1 = 0$ or $x^2 + x + 1 = 0$

$\therefore \quad x = 1$ is the real root. $x^2 + x + 1 = 0$ has two complex roots.

Thus, given x is the complex cube root of unity, it fulfills the equation $x^2 + x + 1 = 0$. More… $x^3 = 1$.

Now, $$AB = \begin{bmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{bmatrix} \cdot \begin{bmatrix} x & x^2 & 1 \\ x^2 & 1 & x \\ 1 & x & x^2 \end{bmatrix}$$

$$= \begin{bmatrix} x + x^3 + x^2 & x^2 + x + x^3 & 1 + x^2 + x^4 \\ x^2 + x^4 + 1 & x^3 + x^2 + x & x + x^3 + x^2 \\ x^3 + x^2 + x & x^4 + 1 + x^2 & x^2 + x + x^3 \end{bmatrix}$$

Now, here write x^4 as $x^3 \cdot x$ and take $x^3 = 1 \quad \therefore \quad x^4 = 1 \cdot x = x$

$$\therefore \quad AB = \begin{bmatrix} x^2 + x + 1 & x^2 + x + 1 & x^2 + x + 1 \\ x^2 + x + 1 & x^2 + x + 1 & x^2 + x + 1 \\ x^2 + x + 1 & x^2 + x + 1 & x^2 + x + 1 \end{bmatrix}$$

But we have $x^2 + x + 1 = 0$

$$\therefore \quad AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Null matrix}$$

(10) If $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, **show that** $A_\alpha \cdot B_\beta = B_\beta \cdot A_\alpha = A_{\alpha+\beta}$

Solution :

Given $$A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \quad \therefore \quad A_\beta = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

and $$A_{\alpha+\beta} = \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$\therefore \quad A_\alpha \cdot B_\beta = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha \cdot \sin\beta & \cos\alpha \cdot \sin\beta + \sin\alpha\cos\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & -\sin\alpha \cdot \sin\beta + \cos\alpha \cdot \cos\beta \end{bmatrix}$$

Rewriting the terms of the matrix so that the trigonometric formulae suit directly, we get

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta \\ -\{\sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta\} & \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$= A_{\alpha+\beta} \qquad \ldots (1)$$

Further

$$A_\beta \cdot A_\alpha = \begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha \cdot \cos\beta - \sin\alpha\sin\beta & \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ -\cos\alpha \cdot \sin\beta - \sin\alpha \cdot \cos\beta & -\sin\alpha \cdot \sin\beta + \cos\alpha \cdot \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$= A_{\alpha+\beta} \qquad \ldots (2)$$

From results (1) and (2), it is clear that

$A_\alpha \cdot A_\beta = A_\beta \cdot A_\alpha = A_{\alpha+\beta}$

(11) If $A = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 7 & -5 \\ 0 & 5 \end{bmatrix}$, **verify that (AB) C = A (BC).**

Solution :

Since A, B, C are square matrices of order 2 × 2, therefore, every product of these three matrices is defined.

$$\therefore \quad AB = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} -2 - 8 & 10 + 6 \\ -3 + 4 & 15 - 3 \end{bmatrix} = \begin{bmatrix} -10 & 16 \\ 1 & 12 \end{bmatrix}$$

$$\therefore \quad (AB)\,C = \begin{bmatrix} -10 & 16 \\ 1 & 12 \end{bmatrix} \cdot \begin{bmatrix} 7 & -5 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -70 + 0 & 50 + 80 \\ 7 + 0 & -5 + 60 \end{bmatrix} = \begin{bmatrix} -70 & 130 \\ 7 & 55 \end{bmatrix} \qquad \ldots (1)$$

Further $$BC = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix} \cdot \begin{bmatrix} 7 & -5 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -7 + 0 & 5 + 25 \\ 28 + 0 & -20 - 15 \end{bmatrix} = \begin{bmatrix} -7 & 30 \\ 28 & -35 \end{bmatrix}$$

$$\therefore \quad A\,(BC) = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -7 & 30 \\ 28 & -35 \end{bmatrix}$$

$$= \begin{bmatrix} -14 - 56 & 60 + 70 \\ -21 + 28 & 90 - 35 \end{bmatrix} = \begin{bmatrix} -70 & 130 \\ 7 & 55 \end{bmatrix} \qquad \ldots (2)$$

From results (1) and (2), it is clear that (AB) C = A (BC)

(12) If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -1 & -1 \\ -2 & 2 & 3 \end{bmatrix}$,

show that A (B + C) = AB + AC.

Solution :

B and C are of same order $\therefore$ (B + C) is defined.

$$\therefore \quad B + C = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 & -1 \\ -2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}_{2 \times 3}$$

A is 2 × 2, (B + C) is 2 × 3 ∴ A (B + C) is 2 × 3

$$\therefore \quad A(B+C) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 & 10 \\ 7 & 4 & 8 \end{bmatrix} \quad \ldots (1)$$

Next AB, AC and AB + AC are defined.

$$\therefore \quad AB + AC = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 & 5 \\ 5 & 4 & 7 \end{bmatrix} + \begin{bmatrix} -2 & 3 & 5 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 & 10 \\ 7 & 4 & 8 \end{bmatrix} \quad \ldots (2)$$

From results (1) and (2), it is clear that $A(B + C) = AB + AC$.

(13) If $f(x) = x^2 - 5x + 6$, evaluate $f(A)$, where $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$.

Solution :

Given

$f(x) = x^2 - 5x + 6$

∴ $f(A) = A^2 - 5A + 6I$ where I is a unit matrix of order 3 × 3.

$$\text{Now, } A^2 = A \cdot A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

Now, $f(A) = A^2 - 5A + 6I$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5 \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10+6 & -1-0+0 & 2-5+0 \\ 9-10+0 & -2-5+6 & 5-15+0 \\ 0-5+0 & -1+5+0 & -2-0+6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

(14) Find the values of x and y, if $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x & 5 & -3 \\ 2 & y & 5 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 7 \\ 7 & 7 & 1 \end{bmatrix}$.

Solution :

$$\text{Given} \quad \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x & 5 & -3 \\ 2 & y & 5 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 7 \\ 7 & 7 & 1 \end{bmatrix}$$

Multiplying matrices on L.H.S., we get

$$\begin{bmatrix} x+4 & 5+2y & -3+10 \\ 3x+4 & 15+2y & -9+10 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 7 \\ 7 & 7 & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x+4 & 5+2y & 7 \\ 3x+4 & 15+2y & 1 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 7 \\ 7 & 7 & 1 \end{bmatrix}$$

By equality of matrices, we get

$x + 4 = 5 \quad \therefore x = 5 - 4 \quad \therefore x = 1$

and $5 + 2y = -3 \quad \therefore 2y = -3 - 5 \quad \therefore 2y = -8 \quad \therefore y = -4$

Thus, $x = 1, \ y = -4$

(15) Find x and y, if $\left\{3 \begin{bmatrix} 4 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix} - 2 \begin{bmatrix} 3 & 2 & 4 \\ -6 & 1 & -3 \end{bmatrix}\right\} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Solution :

$$\text{Given} \quad \left\{3 \begin{bmatrix} 4 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix} - 2 \begin{bmatrix} 3 & 2 & 4 \\ -6 & 1 & -3 \end{bmatrix}\right\} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Upon simplification, we get

$$\therefore \quad \left\{\begin{bmatrix} 12 & 3 & 9 \\ 0 & -3 & -9 \end{bmatrix} - \begin{bmatrix} 6 & 4 & 8 \\ -12 & 2 & -6 \end{bmatrix}\right\} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 6 & -1 & 1 \\ 12 & -5 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Multiplying the matrices on L.H.S. we get

$$\therefore \quad \begin{bmatrix} 6-3-2 \\ 12-15+6 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

By equality of matrices, $x = 1, \ y = 3$

(16) If $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, find 'X' such that AX = I, where I is 2 × 2.

Solution :

Here

$$A_{2 \times 2} \cdot X_{2 \times 2} = I_{2 \times 2}$$

Since A is 2 × 2, I is 2 × 2 ∴ X must be 2 × 2.

$$\text{Let } X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then from $A \cdot X = I$, we get

$$\therefore \quad \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplying the matrices on L.H.S., we get

$$\therefore \quad \begin{bmatrix} 2a + c & 2b + d \\ -c & -d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By equality of matrices,

$2a + c = 1$ and $2b + d = 0$

$-c = 0$; $-d = 1$

$\therefore c = 0$; $\therefore d = -1$

Putting $c = 0$ in $2a + c = 1$; Putting $d = -1$ in $2b + d = 0$

$\therefore 2a = 1$; $\therefore 2b - 1 = 0$

$\therefore a = \frac{1}{2}$; $\therefore 2b = 1$

$\therefore b = \frac{1}{2}$

$$\therefore X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$$

(17) If $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, **show that** $A^2 = A$.

Solution :

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 + 2 - 4 & -4 - 6 + 8 & -8 - 8 + 12 \\ -2 - 3 + 4 & 2 + 9 - 8 & -4 + 12 - 12 \\ 2 + 2 - 3 & -2 - 6 + 6 & -4 - 8 + 9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= A$$

Thus, $A^2 = A$.

(18) Find the unknown matrix 'X', if $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} X = \begin{bmatrix} 6 & 2 & 4 \\ 12 & 4 & 8 \\ 3 & 1 & 2 \end{bmatrix}$.

Solution :

Given $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}_{3 \times 1} \cdot X_{1 \times 3} = \begin{bmatrix} 6 & 2 & 4 \\ 12 & 4 & 8 \\ 3 & 1 & 2 \end{bmatrix}_{3 \times 3}$

$\therefore$ *X must be* 1×3

Let $X = [a \quad b \quad c]$

$$\therefore \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} [a \quad b \quad c] = \begin{bmatrix} 6 & 2 & 4 \\ 12 & 4 & 8 \\ 3 & 1 & 2 \end{bmatrix}$$

Multiplying matrices on L.H.S. we get

$$\therefore \begin{bmatrix} 2a & 2b & 2c \\ 4a & 4b & 4c \\ a & b & c \end{bmatrix} = \begin{bmatrix} 6 & 2 & 4 \\ 12 & 4 & 8 \\ 3 & 1 & 2 \end{bmatrix}$$

By equality of matrices, we get

$a = 3, \ b = 1, \ c = 2$

$$\therefore X = [3 \quad 1 \quad 2]_{1 \times 3}$$

(19) If $A = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -2 & 3 \\ 2 & -2 & 3 \end{bmatrix}$, **show that** $A^2 = I$.

Solution :

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -2 & 3 \\ 2 & -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & -1 \\ 3 & -2 & 3 \\ 2 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 3 - 2 & 0 - 2 + 2 & 0 + 3 - 3 \\ 0 - 6 + 6 & 3 + 4 - 6 & -3 - 6 + 9 \\ 0 - 6 + 6 & 2 + 4 - 6 & -2 - 6 + 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Thus $A^2 = I$

(20) If $A + I = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 1 & 3 \\ -2 & -3 & 1 \end{bmatrix}$, **obtain the matrix** $(A + I)(A - I)$.

Solution :

Given $A + I = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 1 & 3 \\ -2 & -3 & 1 \end{bmatrix}$

$\therefore$ I must be 3×3 and we first find matrix A.

$$\therefore A = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 1 & 3 \\ -2 & -3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & 4 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

$$\therefore A - I = \begin{bmatrix} 0 & 3 & 4 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & 4 \\ -1 & -1 & 3 \\ -2 & -3 & -1 \end{bmatrix}$$

$$\therefore (A + I)(A - I) = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 1 & 3 \\ -2 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 & 4 \\ -1 & -1 & 3 \\ -2 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 - 3 - 8 & 3 - 3 - 12 & 4 + 9 - 4 \\ 1 - 1 - 6 & -3 - 1 - 9 & -4 + 3 - 3 \\ 2 + 3 - 2 & -6 + 3 - 3 & -8 - 9 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -12 & 9 \\ -6 & -13 & -4 \\ 3 & -6 & -18 \end{bmatrix}$$

.... EXERCISES ....

(1) (a) Determine A + BC given that $A = \begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 0 & 4 \end{bmatrix}$.

(b) If $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, then compute $A^2 - 7A$.

(c) If $A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 1 \\ 1 & 2 \end{bmatrix}$,

find the matrix AB − 3I where I is the unit matrix of order 2.

(d) If $A = \begin{bmatrix} 1 & -2 \\ 0 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & 0 \end{bmatrix}$,

find the matrix AB and without computing BA, show that AB ≠ BA.

(e) If $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$, find A (B − C).

(f) If $A = \begin{bmatrix} 1 & 5 & 1 \\ -3 & 2 & 5 \\ 4 & 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 & 4 \\ 6 & 6 & -5 \\ -1 & 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 3 & 3 \\ 1 & 1 & 2 \end{bmatrix}$,

find the matrix A (B − C).

(g) If $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$, find AB and BA. Is AB = BA?

(h) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, find the matrix $A^2 - 5A + 7I$.

(i) Evaluate $A^2 - 3A + 9I$, if $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$.

(j) If $A = \begin{bmatrix} 2 & 5 \\ 6 & 7 \end{bmatrix}$, find $A^2 + 4A + 2I$, where I = unit matrix.

(2) (a) If $A = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}$, show that A^2 is a null matrix.

(b) If $A = \begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix}$, prove that $A^2 = 0$.

(c) If $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, show that $A^2 = 2A$.

(d) If $A = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$, show that $A^2 = A$.

(e) Show that the matrices $A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix}$ are commute.

(f) If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, show that AB = BA.

(g) If $A = \begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$, verify that AB ≠ BA.

(h) If $A = \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}$, show that AB = BA = I.

(i) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$, show that AB = I.

(j) If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, show that AB = I.

(k) If $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$, show that $A^2 - 3A - 2I = 0$, where I is the unit matrix of order 2.

(l) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A$ is a scalar matrix.

(3) (a) If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, show that $A^2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$.

(b) If $A = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 6 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -5 \\ 7 & 8 \\ 1 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & -5 \\ 1 & 1 \end{bmatrix}$,

verify that (AB) C = A (BC).

(c) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$,

verify that (AB) C = A (BC).

(d) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ -1 & 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 7 & -2 \\ 6 & 5 & -4 \\ -2 & 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 5 & -1 & 0 \\ -7 & 3 & 4 \\ 2 & 1 & 7 \end{bmatrix}$,

verify that (AB) C = A (BC).

(e) If $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -1 & -1 \\ -2 & 1 & 3 \end{bmatrix}$,

verify that A (B + C) = AB + AC and A (B − C) = AB − AC.

(f) If $A = \begin{bmatrix} 5 & 4 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$,

verify that (A + B) · C = AC + BC and (A − B) · C = AC − BC.

(g) If $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$,

then verify that AB = AC. What do you conclude from this result?

Note : AB = AC does not imply B = C. Cancellation law does not hold.

(h) Show that the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}$ satisfies $A^2 - 4A + 3I = 0$, where I = unit matrix.

(i) If $A = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$,

verify that $(A + B)(A - B) = A^2 + BA - AB - B^2$.

(j) If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$ verify that (AB) C = A

(4) (a) Find x and y, if $\begin{bmatrix} 3 & 1 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ y \end{bmatrix}$.

(b) Find x and y, if $[x \;\; y] \begin{bmatrix} 3 & 0 \\ -4 & 5 \end{bmatrix} = [-10 \;\; 15]$.

(c) Find the values of x, y, z, if $\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} [1 \;\; 2] \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(d) Find the values of x and y from the matrix equation

$$\left\{ 4 \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 3 \end{bmatrix} - 2 \begin{bmatrix} 1 & 3 & -1 \\ 2 & -3 & 4 \end{bmatrix} \right\} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

(e) Find x, y, z, if $\left\{ \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix} + 2 \begin{bmatrix} 3 & 0 & 2 \\ 1 & 4 & 5 \\ 2 & 1 & 0 \end{bmatrix} \right\} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(f) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -2 & 3 \\ 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

find the values of x, y, z, if $(5A - 3B) \cdot C = X$.

(g) If $A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 4 & -2 \\ 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$,

find the values of a, b and c, if $(2A + 3B) \cdot C = X$.

(h) If $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} x & 0 \\ y & 2 \end{bmatrix}$ and $AB = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$,

find x and y.

(i) If $A = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & x \\ y & 2 \end{bmatrix}$,

find x and y such that AB is a null matrix.

(j) If $A = \begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & x \\ y & 0 \end{bmatrix}$ and $(A + B)(A - B) = A^2 - B^2$, find x

(k) Find the values of 'a' and 'b' from the matrix equation,

$$\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a & 1 \\ 5 & b \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -3 & 5 \end{bmatrix}.$$

(l) Find x, y satisfying the matrix equation,

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x & y & 3 \\ 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 7 \\ 9 & 4 & 13 \end{bmatrix}.$$

(5) (a) If $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -5 & x \\ y & -1 \end{bmatrix}$ and AB = I, where I = Identity matrix, find x

(b) If $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$,

find a matrix 'X' such that AX = B.

(c) Find the matrix 'X', if $A = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 7 \\ 2 & 3 \end{bmatrix}$ and AX = B.

(d) Find the unknown matrix 'X', if $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \cdot X = \begin{bmatrix} 6 & 2 & 4 \\ 12 & 4 & 8 \\ 3 & 1 & 2 \end{bmatrix}$.

(e) Find the matrix 'X' such that AX = B,

if $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

(f) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 0 \\ 3 & 6 \end{bmatrix}$.

find the matrix C such that AC = B.

(g) Solve for 'X' and 'Y', if

$$3X - Y = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } X - 3Y = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

.... ANSWERS ....

(1) (a) $\begin{bmatrix} -2 & 16 \\ 12 & 14 \end{bmatrix}$ (b) $\begin{bmatrix} -10 & -3 \\ 0 & -10 \end{bmatrix}$ (c) $\begin{bmatrix} -4 & 7 \\ 5 & -2 \end{bmatrix}$

(d) $AB = \begin{bmatrix} 1 & -5 & 4 \\ 2 & 4 & 0 \\ 8 & -5 & 12 \end{bmatrix}_{3 \times 3}$, BA is 2×2 $\therefore AB \neq BA$

(e) $\begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}$ (f) $\begin{bmatrix} 9 & 18 & -39 \\ -9 & -3 & -35 \\ 10 & 21 & -12 \end{bmatrix}$ (g) Yes, AB = BA

(h) Zero matrix (i) $\begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$ (j) $\begin{bmatrix} 44 & 65 \\ 78 & 109 \end{bmatrix}$

(4) (a) $x = -1$, $y = -4$ (b) $x = \frac{2}{3}$, $y = 3$ (c) $x = 3$, $y = z = 2$

(d) $x = 2$, $y = 4$ (e) $x = 31$, $y = 53$, $z = 19$ (f) $x = -2$, $y = 8$, $z = -6$

(g) $a = 14$, $b = 8$, $c = 21$ (h) $x = 0$, $y = 1$ (i) $x = -6$, $y = -1$

(j) $x = 2$, $y = 2$ (k) $a = -2$, $b = 1$ (l) $x = 1$, $y = 2$

(5) (a) $x = 3$, $y = 2$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \end{bmatrix}$ (c) $\frac{1}{5} \begin{bmatrix} 6 & 17 \\ 4 & 8 \end{bmatrix}$

(d) $X = [3 \;\; 1 \;\; 2]$ (e) $\frac{1}{10} \begin{bmatrix} 3 \\ -14 \end{bmatrix}$ (f) $C = \frac{1}{7} \begin{bmatrix} 11 & -6 \\ 16 & 18 \end{bmatrix}$

(g) $X = \frac{1}{8} \begin{bmatrix} 3 & 4 \\ -4 & 4 \end{bmatrix}$, $Y = \frac{1}{8} \begin{bmatrix} 1 & 4 \\ -4 & 4 \end{bmatrix}$

4.3.7 Transpose of a Matrix

The matrix obtained from a given matrix A by interchanging the rows and columns is called transpose of A. It is denoted by A^{-1} or A^T.

For instance,

If $A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 6 \end{bmatrix}_{2\times 3}$, then A' or $A^T = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 4 & 6 \end{bmatrix}_{3\times 2}$.

Remarks :

(1) It follows that : $(A')' = A$.

(2) If A is m × n, then A' is n × m. Thus, both the products AA' and A'A exist and are of different orders unless A is a square matrix.

(3) The transpose of a row matrix is a column matrix and vice versa.

(4) The transpose of a scalar quantity is the same scalar quantity.

Properties of Transposition of Matrices :

(1) The transpose of the sum of two matrices is equal to the sum of their transposes. Thus, if A, B are two matrices of the same order, then

$$(A + B)' = A' + B' = B' + A'.$$

(2) The transpose of the product of two matrices is equal to the product of the transposes, if the order of multiplication is reversed. Thus, if A and B are two matrices confirmable for the product AB, then

$$(AB)' = B'A'$$

(3) If $A \cdot A' = A' \cdot A = I$, then A is called orthogonal. If $|A| = 1$, A is called proper matrix and if $|A| = -1$, then A is called improper matrix.

Important Result :

Any square matrix (A) can be expressed as sum of a Symmetric matrix (B) and a Skew-symmetric matrix (C).

Thus, $A = B + C$

where B = Symmetric matrix $= \frac{1}{2}(A + A')$

C = Skew-symmetric matrix $= \frac{1}{2}(A - A')$

$$\therefore \quad A = \underbrace{\frac{1}{2}(A + A')}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(A - A')}_{\text{Skew-symmetric}}$$

.... ILLUSTRATIVE EXAMPLES ....

(1) If $A = \begin{bmatrix} 2 & 5 & 6 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 1 \\ 0 & 4 \\ 5 & 7 \end{bmatrix}$ **verify that** $(AB)' = B'A'$.

Solution :

In this problem,

A is 2 × 3, B is 3 × 2 ∴ AB is defined and is 2 × 2.

$$\therefore \quad AB = \begin{bmatrix} 2 & 5 & 6 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 0 & 4 \\ 5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 12+0+30 & 2+20+42 \\ 0+0+10 & 0+4+14 \end{bmatrix}$$

$$AB = \begin{bmatrix} 42 & 64 \\ 10 & 18 \end{bmatrix}$$

$$\therefore \quad (AB)' = \begin{bmatrix} 42 & 10 \\ 64 & 18 \end{bmatrix} \qquad \ldots (1)$$

Further $A = \begin{bmatrix} 2 & 5 & 6 \\ 0 & 1 & 2 \end{bmatrix}_{2\times 3}$ $\quad \therefore A' = \begin{bmatrix} 2 & 0 \\ 5 & 1 \\ 6 & 2 \end{bmatrix}_{3\times 2}$

and $B = \begin{bmatrix} 6 & 1 \\ 0 & 4 \\ 5 & 7 \end{bmatrix}_{3\times 2}$ $\quad \therefore B' = \begin{bmatrix} 6 & 0 & 5 \\ 1 & 4 & 7 \end{bmatrix}_{2\times 3}$

Since B' is 2 × 3, A' is 3 × 2, ∴ B'A' is 2 × 2

$$\therefore \quad B'A' = \begin{bmatrix} 6 & 0 & 5 \\ 1 & 4 & 7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 5 & 1 \\ 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+0+30 & 0+0+10 \\ 2+20+42 & 0+4+14 \end{bmatrix}$$

$$= \begin{bmatrix} 42 & 10 \\ 64 & 18 \end{bmatrix} \qquad \ldots (2)$$

Form results (1) and (2), we have

$$(AB)' = B'A'$$

Hence, the result.

(2) Express the matrix A as the sum of a symmetric and a skew symmetric matrix, where

$$A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}.$$

Solution :

We know that for a symmetric matrix, A' = A and for a skew-symmetric matrix, A' = –A.

Thus, if A is the given square matrix, then we have :

$$A = \underbrace{\frac{1}{2}(A + A')}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(A - A')}_{\text{Skew-symmetric}} \qquad \ldots (1)$$

Now, Given $A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$ $\quad A' = \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$

$$\therefore \quad A + A' = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix} + \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix} = \begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix} \qquad \ldots (2)$$

and $A - A' = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix} - \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$

Substituting (2) and (3) in expression (1), we get

$$A = \frac{1}{2}\begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 4 & \frac{3}{2} & -4 \\ \frac{3}{2} & 3 & -3 \\ -4 & -3 & -7 \end{bmatrix}}_{\text{Symmetric matrix}} + \underbrace{\begin{bmatrix} 0 & \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}}_{\text{Skew-symmetric matrix}}$$

(3) Show that the matrix $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ is an orthogonal matrix.

Solution :

Given $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

$$A' = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\therefore \quad A \cdot A' = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + 0 + \sin^2\theta & 0+0+0 & -\sin\theta\cos\theta + 0 + \sin\theta\cos\theta \\ 0+0+0 & 0+1+0 & 0+0+0 \\ -\sin\theta\cos\theta + 0 + \sin\theta\cos\theta & 0+0+0 & \sin^2\theta + 0 + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \ldots \because \sin^2\theta + \cos^2\theta = 1$$

$$= I$$

Thus, $AA' = I$

$\therefore$ Matrix A is or thogonal.

.... EXERCISES ....

(1) If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 \\ -2 & 5 \end{bmatrix}$ verify that $(AB)' = B'A'$.

(2) Verify that (a) $(AB)' = B'A'$ (b) $(A + B)' = A' + B'$.

Given that $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$.

(3) If $A = \begin{bmatrix} 3 & 4 \\ -2 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, verify that $(AB)' = B'A'$.

(4) If $A = \begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}$, verify that $(AB)' = B'A'$. [W'07]

(5) If $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 4 & 0 \end{bmatrix}$, verify that $(AB)' = B'A'$.

(6) If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, verify that $AA' = A' \cdot A = I$ where I = Identity matrix.

(7) Verity whether the given matrix A is orthogonal?

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

(8) Express the matrix A as the sum of symmetric and skew-symmetric matrices, where

$$A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$$

(9) If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ verify that $(AB)' = B'A'$. [S'08]

.... ANSWERS ....

(8) $\begin{bmatrix} -1 & \frac{9}{2} & 3 \\ \frac{9}{2} & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{2} & -2 \\ -\frac{5}{2} & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$

3.8 Determinant of a Matrix

If A is any square matrix, then its determinant is written as det (A) or $|A|$. A determinant can be formed keeping the elements in the same positions as the matrix A and replacing the square brackets by vertical brackets.

Thus, if $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, then det (A) or $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$

its value $= (2)(5) - (4)(3) = 10 - 12 = -2$

Singular Matrix : A square matrix A is called a singular matrix, if det (A) or $|A| = 0$.

Non-singular Matrix : A square matrix A is called a non-singular, if $|A| \neq 0$.

Important Property :

$$|AB| = |A| \cdot |B|$$

.... ILLUSTRATIVE EXAMPLES ....

(1) If $A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$ show that the matrix AB is a non-singular.

Solution :

A is 2×3, B is 3×2, $\therefore$ AB is 2×2

$$\therefore \quad AB = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0+0+1 & -2+0+1 \\ 0+4+3 & 1+6+3 \end{bmatrix}$$

$$\therefore \quad AB = \begin{bmatrix} 1 & -1 \\ 7 & 10 \end{bmatrix}$$

$$\therefore \quad |AB| = \begin{vmatrix} 1 & -1 \\ 7 & 10 \end{vmatrix} = (1)(10) - (-7)$$

$$= 10 + 7 = 17 \neq 0$$

$\therefore$ AB is a non-singular matrix.

(2) If $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$ **verify that** $|AB| = |A|\ |B|$

Solution :

$$AB = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3 & 4-2 \\ 0+9 & 0-6 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 9 & -6 \end{bmatrix}$$

$$\therefore \quad |AB| = \begin{vmatrix} 5 & 2 \\ 9 & -6 \end{vmatrix} = (5)(-6) - (9)(2) = -30 - 18 = -48 \quad \ldots (1)$$

Now

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$\therefore \quad |A| = \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = (2)(3) - (0)(1) = 6 - 0 = 6$$

$$B = \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}$$

$$\therefore \quad |B| = \begin{vmatrix} 1 & 2 \\ 3 & -2 \end{vmatrix} = (1)(-2) - (3)(2) = -2 - 6 = -8$$

$$\therefore \quad |A| \cdot |B| = (6)(-8) = -48 \quad \ldots (2)$$

From results (1) and (2), we have :

$$|AB| = |A|\ |B|$$

.... EXERCISES ....

(1) If $A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & -3 \\ 0 & -1 & 1 \end{bmatrix}$, find $|A|$. Is the matrix A a non-singular?

(2) If $A = \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}$, find AB, $|AB|$. Is the matrix AB a non-singular?

(3) If $A = \begin{bmatrix} 3 & 2 & -5 \\ 4 & 5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 \\ -2 & 3 \\ 0 & -1 \end{bmatrix}$, find $|AB|$. Is AB a non-singular matrix?

(4) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, show that AB is non-singular matrix.

(5) If $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & -4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ -3 & 1 \\ 4 & -1 \end{bmatrix}$, find $|AB|$. Is the matrix AB singular?

(6) If $A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -7 & 5 \\ -4 & 3 \end{bmatrix}$, verify that $|AB| = |A|\ |B|$

(7) If $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, find $|A - B|$.

(8) If $A = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$, find $|AA'|$.

.... ANSWERS ....

(1) $|A| = -12 \neq 0$ $\therefore$ Matrix A is non-singular.

(2) $|AB| = 1 \neq 0$ $\therefore$ Matrix AB is a non-singular.

(3) $|AB| = 69$ $\therefore$ AB is a non-singular matrix.

(5) $|AB| = -10$ $\therefore$ AB is a non-singular matrix.

(7) $|A - B| = 0$

(8) 1

4.4 MINOR AND CO-FACTOR OF AN ELEMENT a_{ij} OF A SQUARE MATRIX

Let a square matrix A be $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

$\therefore$ its determinant is $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Now,

(1) In the determinant $|A|$, let the i^{th} row and the j^{th} column be deleted and a new determinant is formed having $(n - 1)$ rows and columns. This new determinant is called the minor of a_{ij} and denoted by M_{ij}.

Thus, in the above determinant $|A|$, the minor of a_{21} is :

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

In the similar way, we can write minors of all elements of $|A|$.

(2) The co-factor C_{ij} of the element a_{ij} is the signed minor of a_{ij} determined by the rule.

$$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

For instance : $C_{21} = (-1)^{2+1} M_{21} = (-1)^3 M_{21} = -M_{21}$

In other words, minor with the corresponding sign of an element a_{ij} in $|A|$ is called its co-factors.

(3) In third order determinant, the pattern of signs for co-factors is as below :

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

(4) Let Δ or $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a 3×3 order determinant, then it can be written in ... order determinant taking minor of each element of either first row or first column or ... as :

$$\Delta = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

This is called the expansion of a determinant first row-wise. With the help of this method of expansion, we can easily find the value of a determinant.

$$\therefore \Delta = a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 a_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$$

= Real number.

4.5 ADJOINT OF A MATRIX

(a) When A is a square matrix of order 2×2 :

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\therefore |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$= (a_{11})(a_{22}) - (a_{21})(a_{12})$

We define,

A_{11} = Co-factor of element a_{11}

$= (-1)^{1+1} M_{11} = M_{11} = a_{22}$

A_{12} = Co-factor of a_{12}

$= (-1)^{1+2} \cdot M_{12} = -M_{12} = -a_{21}$

A_{21} = Co-factor of a_{21}

$= (-1)^{2+1} M_{21} = -M_{21} = -a_{12}$

A_{22} = Co-factor of a_{22}

$= (-1)^{2+2} M_{22} = M_{22} = a_{11}$

Then, matrix of co-factors = $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

Now, the transpose of the matrix (A_{ij}) i.e. co-factors is called the adjoint of the matrix A and is denoted by adj A.

$\therefore$ adj A = Transpose of the above matrix of co-factors

$$= \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

(b) When A is a square matrix of order 3×3 :

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ $\therefore |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

If A_1, B_1, C_1 ... are co-factors of a_1, b_1, c_1 ... in the determinant |A|, then the matrix of co-factors is

$$= \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

= P, say.

$\therefore$ adj A = Transpose of P

$$= \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

.... ILLUSTRATIVE EXAMPLES ....

(1) If $A = \begin{bmatrix} 6 & 5 \\ 2 & 1 \end{bmatrix}$, find adj A.

Solution :

Given $A = \begin{bmatrix} 6 & 5 \\ 2 & 1 \end{bmatrix}$ $\therefore |A| = \begin{vmatrix} 6 & 5 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} + & - \\ - & + \end{vmatrix}$

$= (6)(1) - (2)(5)$

$= 6 - 10 = -4 \neq 0$

Now A_{11} = Co-factor of element 6

= +1

A_{12} = Co-factor of element 5

= −2

A_{21} = Co-factor of element 2

= −5

A_{22} = Co-factor of element 1

= 6

$\therefore$ P = Matrix of co-factors of elements in | A|

$$= \begin{bmatrix} 1 & -2 \\ -5 & 6 \end{bmatrix}$$

$\therefore$ adj A = Transpose of Matrix P

$$= \begin{bmatrix} 1 & -5 \\ -2 & 6 \end{bmatrix}$$

(2) Compute adjoint of matrix : $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$.

Solution :

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\therefore |A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Then A_{11} = Co-factor of a_{11} i.e. 1st row, 1st column element.

$= (-1)^{1+1} M_{11} = +\begin{vmatrix} 3 & 5 \\ 5 & 12 \end{vmatrix} = +(36 - 25) = 11$

$A_{12} = -\begin{vmatrix} 1 & 5 \\ 1 & 12 \end{vmatrix} = -(12 - 5) = -7$

$A_{13} = +\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = +(5 - 3) = +2$

$A_{21} = -\begin{vmatrix} 2 & 3 \\ 5 & 12 \end{vmatrix} = -(24 - 15) = -9$

$A_{22} = +\begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = +(12 - 3) = 9$

$A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = -(5 - 2) = -3$

$A_{31} = +\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = +(10 - 9) = 1$

$A_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = -(5 - 3) = -2$

$A_{33} = +\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = +(3 - 2) = 1$

$\therefore$ The matrix of co-factors $= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 11 & -7 & 2 \\ -9 & 9 & -3 \\ 1 & -2 & 1 \end{bmatrix}$

Hence, adj A = Transpose of matrix of co-factors

$= \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

.... EXERCISES ....

(1) Find adjoint matrix in each of the following matrices :

(a) $\begin{bmatrix} 3 & 4 \\ 0 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 3 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ (g) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

(2) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$, compute adjoint A and verify that A (adj A) = |A| I.

(3) Given the matrix $A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. Compute adjoint A and prove that A (adj A) = (ad

= |A| I.

.... ANSWERS ....

(1) (a) $\begin{bmatrix} 7 & -4 \\ 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & -12 & 8 \\ -2 & 6 & -4 \\ 1 & -9 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} -13 & -2 & 7 \\ 3 & 1 & -2 \\ 2 & 0 & -1 \end{bmatrix}$ (e) $\begin{bmatrix} 7 & 13 & -4 \\ 4 & 14 & 1 \\ 8 & 5 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

4.6 INVERSE OF A MATRIX BY ADJOINT METHOD

If A and B are two square matrices of the same order such that AB = BA = I, where I is the unit matrix, then the matrices A and B are called inverse of each other.

For instance,

$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = I$ and $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = I$

$\therefore$ $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ are inverse of each other.

Remarks :

(1) The inverse of a matrix A is usually denoted by A^{-1}.

(2) If the inverse of a matrix A exists (i.e. A^{-1} exists), then $A^{-1} \cdot A = A \cdot A^{-1} = I$.

(3) A square matrix does not necessarily have an inverse. A square matrix A has an inverse, if and only if A is a non-singular i.e. $|A| \neq 0$.

(4) If the inverse of a matrix exists, then it is unique i.e. it has only one inverse.

(5) For a non-singular square matrix $(A^{-1})^{-1} = A$.

(6) For a square matrix A, $(A')^{-1} = (A^{-1})'$ if $|A| \neq 0, |A^{-1}| \neq 0$.

(7) For non-singular matrices A, B of same order $(AB)^{-1} = B^{-1} A^{-1}$.

Now if $|A| \neq 0$ i.e. if the matrix A is a non-singular, then we have the following important result :

$A \cdot \text{adj } A = \text{adj } A \cdot A = |A| I$ ... (1)

Moreover, $A \cdot A^{-1} = A^{-1} \cdot A = I$ ... (2)

Substituting for I in result (1), we get

$A \cdot \text{adj } A = |A| A A^{-1}$

$\therefore$ $\text{adj } A = |A| A^{-1}$

$\therefore$ $A^{-1} = \frac{1}{|A|} \cdot \text{adj } A$

This is the required result for A^{-1} by adjoint method.

.... ILLUSTRATIVE EXAMPLES ....

(1) Find the inverse of a matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ using the relation $A^{-1} = \frac{1}{|A|} \cdot \text{adj } A$.

Solution :

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix}$$

$= 1(24 - 25) - 2(12 - 15) + 3(10 - 12)$

$= 1(-1) - 2(-3) + 3(-2)$

$= -1 + 6 - 6$

$= -1 \neq 0 \quad \therefore$ A is a non-singular $\therefore$ A^{-1} exists.

To find adj A :

Let $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, say.

Finding co-factors of elements of $|A|$, we have :

A_1 = Co-factor of a_1 (i.e. of element 1)

$= +\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} = +(24 - 25) = -1$

In the same way, we have

$B_1 = -\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} = -(12 - 15) = +3$

$C_1 = +\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = +(10 - 12) = -2$

$A_2 = -\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -(12 - 15) = 3$

$B_2 = +\begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} = +(6 - 9) = -3$

$C_2 = -\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = -(5 - 6) = 1$

$A_3 = +\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = +(10 - 12) = -2$

$B_3 = -\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = -(5 - 6) = 1$

$C_3 = +\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = +(4 - 4) = 0$

$\therefore$ The matrix of co-factors $= \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$

$\therefore$ adj A = Transpose of matrix of co-factors

$= \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \frac{1}{-1} \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

(2) Find A^{-1} by adjoint method, if $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$.

Solution :

Given $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ say}$$

Expanding the determinant first row-wise :

$|A| = 2(4 - 3) - 3(2 - 9) + 1(1 - 6)$

$= 2 + 21 - 5$

$= 18 \neq 0$

Finding co-factors of elements of $|A|$, we have :

$A_1 = +\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = +(4 - 3) = 1$

$B_1 = -\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -(2 - 9) = 7$

$C_1 = +\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = +(1 - 6) = -5$

$A_2 = -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -(6 - 1) = -5$

$B_2 = +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = +(4 - 3) = 1$

$C_2 = -\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = -(2 - 9) = 7$

$A_3 = +\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = +(9 - 2) = 7$

$B_3 = -\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = -(6 - 1) = -5$

$C_3 = +\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = +(4 - 3) = 1$

$\therefore$ Matrix of co-factors $= \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} 1 & 7 & -5 \\ -5 & 1 & 7 \\ 7 & -5 & 1 \end{bmatrix}$

$\therefore$ adj A = Transpose of above matrix of co-factors

$$= \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

Then, according to the adjoint method

$$\therefore \; A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \frac{1}{18} \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

.... EXERCISES ....

(1) Find the inverse of the following matrices by adjoint method :

(a) $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ 5 & -9 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 4 \\ 1 & -1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & -1 \\ -2 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

.... ANSWERS ....

(1) (a) $\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -9 & 2 \\ -5 & 1 \end{bmatrix}$ (c) $\frac{1}{5}\begin{bmatrix} 4 & 1 & -4 \\ 3 & 2 & -8 \\ -1 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$ (g) $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

4.7 EQUATION IN THE MATRIX FORM

Consider the matrix equation as

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

This gives $\begin{bmatrix} a_1 x + b_1 y \\ a_2 x + b_2 y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

By equality of matrices, we get

$a_1 x + b_1 y = c_1$

$a_2 x + b_2 y = c_2$, a set of simultaneous equations in two unknowns.

The converse in true.

For instance, the set of equations $2x + 3y = 5$

$3x - y = 2$

can be written in matrix form as :

$$\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Similarly, consider the matrix equation as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

This gives $\begin{bmatrix} a_1 x + b_1 y + c_1 z \\ a_2 x + b_2 y + c_2 z \\ a_3 x + b_3 y + c_3 z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

By equality of matrices, we get

$a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$

$a_3 x + b_3 y + c_3 z = d_3$, a set of simultaneous equations in three variables (unknowns).

The converse is true.

For instance, the set of equations

$2x - 4y + 3z = 1$

$x - 2y + 4z = 3$

$3x - y + 5z = 2$ can be written in matrix form as :

$$\begin{bmatrix} 2 & -4 & 3 \\ 1 & -2 & 4 \\ 3 & -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

4.8 SOLUTION OF SIMULTANEOUS EQUATIONS BY MATRIX INVERSION METHOD

The set of equations

$a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$

$a_3 x + b_3 y + c_3 z = d_3$

can be written in matrix equation as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

OR $A \cdot X = B$ ... (1)

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ = Matrix of coefficients of x, y, z.

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ = Column matrix of unknowns.

$B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ = Column matrix of constants.

If A is a non-singular matrix i.e. $|A| \neq 0$, then pre-multiplying both sides of (1) by A^{-1}

$$A^{-1} \cdot A \cdot X = A^{-1} B$$

$$\therefore \quad I X = A^{-1} B \quad \because \quad A^{-1} \cdot A = I$$

$$\therefore \quad X = A^{-1} B \quad \because \quad I \cdot X = X$$

$$\therefore \quad X = \frac{1}{|A|} \text{adj} A \cdot B$$

Finally by equality of matrices, we find the values of unknowns x, y and z.

This method of finding unknowns is called the matrix inversion method or solution of the coefficient matrix.

.... ILLUSTRATIVE EXAMPLES ....

(1) Find the inverse of the coefficient matrix of the equations $2x + 5y = 9$, $x + 3y$ ... solve the equations.

Solution :

The equations are

$$2x + 5y = 9$$

and

$$x + 3y = 5$$

Writing them in matrix form, we have :

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ = coefficient matrix

$$\therefore \quad |A| = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} = (2)(3) - (1)(5) = 6 - 5 = 1 \neq 0$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

$\therefore$ adj A = Interchange elements in leading diagonal of A and change the sign in reverse diagonal of A.

$$= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{adj} A = \frac{1}{1} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Next $X = A^{-1} \cdot B$

$$= \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 27 - 25 \\ -9 + 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

By equality of matrices, $x = 2$, $y = 1$.

This is the required solution.

(2) Find the inverse of the coefficient matrix of the equations :

$2x + 3y - z + 3 = 0$, $5x + y + 3z = 10$, $4x + 3y - 2z + 3 = 0$ and hence solve them.

Solution :

The set of equations is, after rewriting,

$$2x + 3y - z = -3$$
$$5x + y + 3z = 10$$
$$4x + 3y - 2z = -3$$

Writing them in matrix form, we get

$$\begin{bmatrix} 2 & 3 & -1 \\ 5 & 1 & 3 \\ 4 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 10 \\ -3 \end{bmatrix}$$

Here let $A = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 1 & 3 \\ 4 & 3 & -2 \end{bmatrix}$ = Matrix of coefficients

$$\therefore \quad |A| = \begin{vmatrix} 2 & 3 & -1 \\ 5 & 1 & 3 \\ 4 & 3 & -2 \end{vmatrix} = \text{Determinant of matrix A.}$$

$$= 2(-2 - 9) - 3(-10 - 12) - 1(15 - 4)$$
$$= -22 + 66 - 11$$
$$= -33 + 66$$
$$= 33 \neq 0 \quad \therefore A^{-1} \text{ exists.}$$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ = Matrix of unknowns $B = \begin{bmatrix} -3 \\ 10 \\ -3 \end{bmatrix}$ = Matrix of constants

To find A^{-1} :

Let $|A| = \begin{vmatrix} 2 & 3 & -1 \\ 5 & 1 & 3 \\ 4 & 3 & -2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, say.

If $A_1, B_1, C_1, \ldots$ are co-factors of elements $a_1, b_1, c_1, \ldots$, then we have :

$$A_1 = + \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} = -2 - 9 = -11$$

$$B_1 = - \begin{vmatrix} 5 & 3 \\ 4 & -2 \end{vmatrix} = -(-10 - 12) = 22$$

$$C_1 = + \begin{vmatrix} 5 & 1 \\ 4 & 3 \end{vmatrix} = +(15 - 4) = 11$$

$$A_2 = - \begin{vmatrix} 3 & -1 \\ 3 & -2 \end{vmatrix} = -(-6 + 3) = 3$$

$$B_2 = + \begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} = +(-4 + 4) = 0$$

$$C_2 = - \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = -(6 - 12) = +6$$

$$A_3 = +\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = +(9+1) = 10$$

$$B_3 = -\begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix} = -(6+5) = -11$$

$$C_3 = +\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = +(2-15) = -13$$

$\therefore$ The matrix of co-factors $= \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} -11 & 22 & 11 \\ 3 & 0 & 6 \\ 10 & -11 & -13 \end{bmatrix}$

$\therefore$ adj A = Transpose of matrix of co-factors

$$= \begin{bmatrix} -11 & 3 & 10 \\ 22 & 0 & -11 \\ 11 & 6 & -13 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \frac{1}{33}\begin{bmatrix} -11 & 3 & 10 \\ 22 & 0 & -11 \\ 11 & 6 & -13 \end{bmatrix}$$

Next, $X = A^{-1} \cdot B$ gives

$$X = \frac{1}{33}\begin{bmatrix} -11 & 3 & 10 \\ 22 & 0 & -11 \\ 11 & 6 & -13 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 10 \\ -3 \end{bmatrix}$$

$$= \frac{1}{33}\begin{bmatrix} 33 + 30 - 30 \\ -66 + 0 + 33 \\ -33 + 60 + 39 \end{bmatrix}$$

$$= \frac{1}{33}\begin{bmatrix} 33 \\ -33 \\ 66 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

By equality of matrices, $x = 1,\ y = -1,\ z = 2.$

This is the required solution.

(3) Solve by matrix method the set of equations :

$$2x + y = 3, \quad 2y + 3z = 4, \quad 2z + 2x = 8.$$

Solution :

Rewriting the set of equations, we get

$$2x + y + 0z = 3$$
$$0x + 2y + 3z = 4$$
$$2x + 0y + 2z = 8$$

Writing in matrix form, we get

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$

$$\therefore |A| = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 3 \\ 2 & 0 & 2 \end{vmatrix} = 2(4-0) - 1(0-6) + 0 = 8 + 6 = 14 \neq 0$$

To find A^{-1} :

Let $|A| = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 3 \\ 2 & 0 & 2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, say.

If $A_1, B_1, C_1, \ldots$ are co-factors of elements $a_1, b_1, c_1, \ldots$ respectively, then :

$$A_1 = +\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = 4 - 0 = 4$$

$$B_1 = -\begin{vmatrix} 0 & 3 \\ 2 & 2 \end{vmatrix} = -(0-6) = 6$$

$$C_1 = +\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = +(0-4) = -4$$

$$A_2 = -\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = -(2-0) = -2$$

$$B_2 = +\begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix} = +(4-0) = 4$$

$$C_2 = -\begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} = -(0-2) = 2$$

$$A_3 = +\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = +(3-0) = 3$$

$$B_3 = -\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = -(6-0) = -6$$

$$C_3 = +\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = +(4-0) = 4$$

$\therefore$ The matrix of co-factors $= \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} 4 & 6 & -4 \\ -2 & 4 & 2 \\ 3 & -6 & 4 \end{bmatrix}$

$\therefore$ adj A = Transpose of co-factors matrix

$$= \begin{bmatrix} 4 & -2 & 3 \\ 6 & 4 & -6 \\ -4 & 2 & 4 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \frac{1}{14}\begin{bmatrix} 4 & -2 & 3 \\ 6 & 4 & -6 \\ -4 & 2 & 4 \end{bmatrix}$$

Next, $X = A^{-1}$ B gives

$$\therefore X = \frac{1}{14}\begin{bmatrix} 4 & -2 & 3 \\ 6 & 4 & -6 \\ -4 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 12 - 8 + 24 \\ 18 + 16 - 48 \\ -12 + 8 + 32 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 28 \\ -14 \\ 28 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$\therefore \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

By equality of matrices, we get $x = 2,\ y = -1,\ z = 2$.

This is the required solution.

.... EXERCISES ....

(1) Find the inverse of the matrix $\begin{bmatrix} 1 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & -3 & 6 \end{bmatrix}$ and hence solve the equations :

$x + 3y + 2z = 6,\ 3x - 2y + 5z = 5,\ 2x - 3y + 6z = 7$

(2) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 4 \\ 1 & 3 & 4 \end{bmatrix}$ and hence solve the equations :

$x + 3y + 3z = 12,\ x + 4y + 4z = 15,\ x + 3y + 4z = 13$

(3) Express the following sets of equations in the matrix form and hence solve them by inverse of the coefficient matrix :

(a) $3x + 2y = 7$, $4x + 3y = 10$

(b) $6x - 3y + 1 = 0$, $11x - 5y + 2 = 0$

(c) $x + y + z = 3$, $3x - 2y + 3z = 4$, $5x + 5y + z = 11$

(d) $x + y + z = 6$, $3x - y + 3z = 10$, $5x + 5y - 4z = 3$

(e) $x + y + z = 2$, $y + z = 1$, $z + x = 3$

(f) $4x - 3y + z = 1$, $x + 4y - 2z = 10$, $2x - 2y + 3z = 4$

(4) Using matrix method, solve the following equations :

$x + 3y + 3z = 12,\ x + 4y + 4z = 15,\ x + 3y + 4z = 13$

.... ANSWERS ....

(1) $x = -1,\ y = 1,\ z = 2$

(2) $x = 3,\ y = 2,\ z = 1$

(3) (a) $x = 1,\ y = 2$ (b) $x = y = -\frac{1}{3}$ (c) $x = y = z = 1$

(d) $x = 1,\ y = 2,\ z = 3$ (e) $x = 1,\ y = -1,\ z = 2$ (f) $x = 2,\ y = 3,\ z = 2$

(4) $x = 3,\ y = 2,\ z = 1$

●●● PROBLEMS OF BOARD PAPERS ●●●

(1) **Find adjoint of matrix A if** $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$. **[W'07, S'08, Marks 4]**

Ans. Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Let $A_1, B_1, C_1, \ldots$ be co-factors of elements $a_1, b_1, c_1, \ldots$ respectively. Remember the signs of elements in the determinant while finding co-factors of elements

$$A_1 = +\begin{vmatrix} 4 & 5 \\ -6 & -7 \end{vmatrix} = -28 + 30 = 2$$

$$B_1 = -\begin{vmatrix} 3 & 5 \\ 0 & -7 \end{vmatrix} = -(-21 - 0) = 21$$

$$C_1 = +\begin{vmatrix} 3 & 4 \\ 0 & -6 \end{vmatrix} = -18 - 0 = -18$$

$$A_2 = -\begin{vmatrix} 0 & -1 \\ -6 & -7 \end{vmatrix} = -(0 - 6) = 6$$

$$B_2 = +\begin{vmatrix} 1 & -1 \\ 0 & -7 \end{vmatrix} = -7 - 0 = -7$$

$$C_2 = -\begin{vmatrix} 1 & 0 \\ 0 & -6 \end{vmatrix} = -(-6 - 0) = 6$$

$$A_3 = +\begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} = 0 + 4 = 4$$

$$B_3 = -\begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} = -(5 + 3) = -8$$

$$C_3 = +\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = +(4 - 0) = 4$$

Therefore, the matrix of co-factors $= \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}$

Hence,

adj A = Transpose of matrix of co-factors

$$= \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

(2) **If** $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$**, prove that** $A^2 - 3A = 2I$**, where 'I' is unit matrix of order two.** **[W'07, Marks 4]**

Ans. Solved Problem (7) on Page 4.21. Practice yourself.

(3) If $A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ verify that $(AB)' = B'A'$. [W...

Solution :

A is 2×3, B is 2×3, $\therefore$ AB is 2×3 (order of product; equal)

$$\therefore \quad AB = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \\ 8 & -1 & 7 \end{bmatrix}$$

$$\therefore \quad (AB)' = \begin{bmatrix} 3 & 8 \\ -2 & -1 \\ 1 & 7 \end{bmatrix} \qquad \dots (1)$$

A' is 2×2, B' is 3×2

$\therefore$ $B'A'$ is 3×2

$$\therefore \quad B'A' = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ -2 & -1 \\ 1 & 7 \end{bmatrix} \qquad \dots (2)$$

Comparing the results (1) and (2), we have

$$(AB)' = B'A'$$

(4) Using matrix method, solve the following equations :

$$x + 3y + 3z = 12; \quad x + 4y + 4z = 15; \quad x + 3y + 4z = 13.$$ [W'07,

Ans. It is similar to solved Problem (3) on Page 4.48. Practice yourself.

The answer is $x = 3,\ y = 2,\ z = 1$.

(5) If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$ verify that $(AB)C = A(BC)$. [S'08,

Ans. Problem-3 (j) of Exercises on Page 4.30. It is exactly similar to solved Problem ... Page 4.23. Practice yourself.

(6) If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, verify that $(AB)' = B'A'$. [S'08,

Ans. Problem (9) of Exercises on Page 4.35. Practice yourself. It is exactly similar to solved ... (1) on Page 4.32.

(7) Using matrix inversion method, solve the equation :

$$x + y + z = 3, \quad x + 2y + 3z = 4; \quad x + 4y + 9z = 6.$$ [S'08,

Ans. The set of equations is

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

Writing them in matrix form, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Let $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} = \text{Matrix of co-efficients}$$

$$\therefore \quad |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= 1(18 - 12) - 1(9 - 3) + 1(4 - 2)$$
$$= 6 - 6 + 2 = 2 \neq 0$$

$\therefore$ A^{-1} exists.

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \text{Matrix of variables}, \quad B = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} = \text{Matrix of constants}$$

To find A^{-1} :

Let $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, say

Let A_1, B_1, C_1 ... be co-factors of elements a_1, b_1, c_1 ... respectively. Then

$$A_1 = +\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = 18 - 12 = 6$$

$$B_1 = -\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -(9 - 3) = -6$$

$$C_1 = +\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 4 - 2 = 2$$

$$A_2 = -\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -(9 - 4) = -5$$

$$B_2 = +\begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = 9 - 1 = 8$$

$$C_2 = -\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -(4 - 1) = -3$$

$$A_3 = +\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1$$

$$B_3 = -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -(3 - 1) = -2$$

$$C_3 = +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$$

$$\therefore \quad \text{Matrix of co-factors} = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$\therefore$ Adj A = Transpose of matrix of co-factors

$$= \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

Now $A^{-1} = \frac{1}{|A|} \cdot \text{adj } A = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

Further, $X = A^{-1} \cdot B$ gives

$$X = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 18 - 20 + 6 \\ -18 + 32 - 12 \\ 6 - 12 + 6 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

By equality of matrices, $x = 2,\ y = 1,\ z = 0$.

(8) If $A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix}$, find $A^2 - 3I$. [S'09, ...

Ans. $A^2 = A \cdot A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -3 & 10 \\ 2 & 3 & 14 \\ 0 & -2 & 10 \end{bmatrix}$

A is 3×3 $\therefore$ I is 3×3

$$A^2 - 3I = \begin{bmatrix} 8 & -3 & 10 \\ 2 & 3 & 14 \\ 0 & -2 & 10 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -3 & 10 \\ 2 & 0 & 14 \\ 0 & -2 & 7 \end{bmatrix}$$

(9) If $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, verify that $(AB)^T = B^T A^T$. [S'09, M...

Ans. Practice yourself as usual. It is similar to solved Problem (1) on Page 4.32.

(10) Find the adjoint of matrix A, if $A = \begin{bmatrix} 1 & 2 & 6 \\ 7 & 2 & 5 \\ 8 & 2 & 10 \end{bmatrix}$. [S'09, M...

Ans. Practice yourself as given in solved Problem (1) above. The answer is $\begin{bmatrix} 10 & -8 & -2 \\ -30 & -38 & 37 \\ -2 & 14 & -12 \end{bmatrix}$

(11) Solve by matrix method the set of equations :

$x + y + z = 2,\ \ y + z = 1,\ \ z + x = 3.$ [S'09, Marks 4]

Ans. It is similar to solved Problem (3) on Page 4.48. Practice yourself.

The answer is $x = 1,\ y = -1,\ z = 2$.

(12) If $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ 3 & 2 \\ 4 & -2 \end{bmatrix}$, verify that $A + B = B + A$. [W'09, Marks 2]

Ans. Solved Problem (11) on Page 4.11.

(13) If $A = \begin{bmatrix} 2 & 5 \\ 6 & 7 \end{bmatrix}$, find $A^2 + 4A + 2I$ where I is unit matrix. [W'09, Marks 4]

Ans. It is similar to solved Problem (8) on Page 4.21. Practice yourself. The answer is $\begin{bmatrix} 44 & 65 \\ 78 & 109 \end{bmatrix}$.

It is a Problem-1 (j) of Exercises on Page 4.28.

(14) Find x and y, if

$$\left\{4\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 3 & -1 \\ 2 & -3 & 4 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$ [W'09, Marks 4]

Ans. It is a Problem-4 (d) of Exercises on Page 4.30. Practice yourself. The answer is $x = 2,\ y = 4$.

(15) Express the matrix A as sum of symmetric and skew-symmetric matrices, where

$$A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}.$$ [W'09, Marks 4]

Ans. Problem (8) of Exercises on Page 4.35. The answer is $\begin{bmatrix} -1 & \frac{9}{2} & 3 \\ \frac{9}{2} & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{2} & -2 \\ -\frac{5}{2} & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$.

It is similar to solved Problem (2) on Page 4.33. Practice yourself.

(16) Find the inverse of $\begin{bmatrix} 1 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & -3 & 6 \end{bmatrix}$ by adjoint method and hence solve the equations

$x + 3y + 2z = 6,\ \ 3x - 2y + 5z = 5,\ \ 2x - 3y + 6z = 7.$ [W'09, Marks 4]

Ans. Problem (1) of Exercises on Page 4.50. The answer is $x = -1,\ y = 1,\ z = 2$. First find adjoint as shown in solved Problem (1) above. It is similar to solved Problem (2) on Page 4.47.

(17) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 0 & -1 \\ 2 & 3 & 1 \end{bmatrix}$, find $2A - 3B$. [S'10, Marks 2]

Ans. Solved Problem (2) of Illustrative Examples on Page 4.8. Practice yourself as usual.

The answer is $\begin{bmatrix} -4 & 4 & -3 \\ -12 & 8 & 13 \\ 8 & 7 & 15 \end{bmatrix}_{3 \times 3}$

(18) *Find the adjoint of the matrix* $A = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 2 & 1 \end{bmatrix}$.

Ans. It is similar to solved Problem (2) of Illustrative Examples on Page 4.39. Practice you

answer is $\begin{bmatrix} -4 & 1 & 0 \\ 10 & -4 & -6 \\ -8 & -5 & -6 \end{bmatrix}$.

(19) If $A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ verify that $(AB)^T = B^T A^T$.

Ans. *It is similar to Problem (3) on Page 4.52. Practice yourself in the similar way.*

(20) *Find the inverse of the matrix* $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ **using adjoint matrix.**

Ans. *Problem-1 (e) of Exercises on Page 4.44. Practice yourself. The answer is* $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

(21) **Solve the equations using matrix method :**

$x + y + z = 3; \quad x + 2y + 3z = 4; \quad x + 4y + 9z = 6.$

Ans. *It is similar to solved Problem (3) of Illustrative Examples on Page 4.48. Practice yourself*

The answer is $x = 2,\ y = 1$ *and* $z = 0$.

Chapter 5

BINOMIAL THEOREM

5.1 INTRODUCTION

An expression containing two terms only is called a **binomial expression.** For instance, $x + 2$, $x + \frac{1}{x}$, $x^2 + \frac{1}{x}$, ... are binomial expressions. After expanding the binomial expressions, we get more than two terms. For instance,

$$(x + a)^2 = x^2 + 2ax + a^2$$

$$(x + a)^3 = x^3 + 3x^2 a + 3xa^2 + a^3$$

In this chapter, we will study the expansion of $(x + a)^n$, whatever the 'n' may be, by the theorem known as **Binomial Theorem.** This Theorem was established by Sir Issac Newton. For this, the knowledge of factorial notations, permutations, combinations is very essential.

5.2 BASIC CONCEPTS USEFUL IN BINOMIAL EXPANSION

While expanding binomial expressions raised to positive integer (n), we use Binomial Theorem. This is associated with the expressions of combinations which in turn require the knowledge of Factorial Notations.

5.2.1 Factorial Notation

The continuous product of the first 'n' natural numbers is called 'factorial n' or 'n factorial' and is denoted by $\lfloor n$ or n !.

Thus, $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots (n - 1) \cdot n$

$\therefore \quad 5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

$3! = 1 \cdot 2 \cdot 3 = 6$