BINOMIAL THEOREM

$$(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \cdots + {}^nC_r x^{n-r} a^r + \cdots + {}^nC_n a^n$$

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_r x^r + \cdots + {}^nC_n x^n$$

$(a+b) = a + b$ 1 1

$(a+b)^2 = a^2 + 2ab + b^2$ 1 2 1

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ 1 3 3 1

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ 1 4 6 4 1

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$ 1 5 10 10 5 1

Ex! — $(1-2x)^5$

$1 + 5\,1^4(-2x) + 10(1)^3(-2x)^2 + 10(1)^2(-2x)^3 + 5(1)(-2x)^4 + (-2x)^5$.

$\Rightarrow 1 - 10x + 40x^2 - 80x^3 + 80x^4 + 32x^5$.

Ex! — $(\sqrt3+\sqrt2)^4 - (\sqrt3-\sqrt2)^4$

$\Rightarrow (\sqrt3)^4 + 4(\sqrt3)^3(\sqrt2) + 6(\sqrt3)^2(\sqrt2)^2 + 4(\sqrt3)(\sqrt2)^3 + (\sqrt2)^4 - \left[(\sqrt3)^4 + 4(\sqrt3)^3(-\sqrt2) + 6(\sqrt3)^2(-\sqrt2)^2 + 4(\sqrt3)(-\sqrt2)^3 + (\sqrt2)^4\right]$

$\Rightarrow \cancel{(\sqrt3)^4} + 4(\sqrt3)^3(\sqrt2) + \cancel{6(\sqrt3)^2(\sqrt2)^2} + 4\sqrt3(\sqrt2)^3 + \cancel{(\sqrt2)^4} - \cancel{(\sqrt3)^4} + 4(\sqrt3)^3(\sqrt2) \cancel{- 6(\sqrt3)^2(\sqrt2)^2} + 4(\sqrt3)(\sqrt2)^3 - \cancel{(\sqrt2)^4}$

$12\sqrt6 + 12\sqrt6 + 8\sqrt6 + 8\sqrt6$

$= 24\sqrt6 + 16\sqrt6$

$\sqrt6(40) = 40\sqrt6$.

General term —

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^{n-r} y^r$$

$$= {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + \dots + {}^nC_{n-1} x^1 y^{n-1} + {}^nC_n x^0 y^n$$

1st term $= {}^nC_0 x^n y^0$

2nd term $= {}^nC_1 x^{n-1} y^1$

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n^{th} term $= {}^nC_{n-1} x^1 y^{n-1}$

$(n+1)^{th}$ term $= {}^nC_n x^0 y^n$.

So, the $(r+1)^{th}$ term (or general term) of the binomial expansion of $(x+y)^n$ ...

$$\boxed{T_{r+1} = {}^nC_r \, x^{n-r} y^r.}$$

* Middle term of the Binomial Expansion

$$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + \dots + {}^nC_{n-1} x^1 y^{n-1} + {}^nC_n x^0 y^n$$

→ The number of terms in the binomial expansion of $(x+y)^n = n+1$.

case-I : n is even (i.e $n+1$ is odd)

middle term $= \left(\frac{n}{2}+1\right)^{th}$

case-II : n is odd.

middle term $= \frac{1}{2}(n+1)^{th}$ term & $\left\{\frac{(n+1)}{2}+1\right\}^{th}$ term

p^{th} term from the end is $(a+b)^n$.

$= (n+1-p+1)^{th}$ term from the begning

$= (n-p+2)^{th}$ term from the begning.

Ex!→ middle term in the Expansion of $\left(\frac{x}{3}+9y\right)^{10}$

$n = 10$, even

sol^n!→ middle term $= \left(\frac{n}{2}+1\right)^{th}$

$= \left(\frac{10}{2}+1\right)^{th} = 6^{th}$ term

$T_{nth} = T_{r+1} = {}^nC_r\, x^{n-r} y^r$

$T_6 = T_{5+1} = {}^{10}C_5 \left(\frac{x}{3}\right)^{10-5}(9y)^5$

$= \frac{10!}{5!(10-5)!}\left(\frac{x}{3}\right)^5(9y)^5$

$= \frac{10!}{5!\,5!}\left(\frac{x^5}{3^5}\right)(9^5)(y^5)$

$= 61236\, x^5 y^5$

Ex!- 5^{th} term from the end $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^9$.

Sol^n!- p^{th} term from the end = $(n-p+2)^{th}$ term from the begining.

5^{th} term from the end = $(9-5+2)^{th} = 6$

$$T_6 = T_{5+1} = {}^9C_5 \left(\frac{x^3}{2}\right)^{9-5} \left(\frac{-2}{x^2}\right)^5$$

$$= \frac{9!}{5!(9-5)!} \times \left(\frac{x^3}{2}\right)^4 \times \left(\frac{-2}{x^2}\right)^5$$

$$= \frac{9!}{5! \times 4!} \times \frac{x^{12}}{2^4} \times \frac{(-2)^5}{x^{10}}$$

$$= \frac{9 \times 8 \times 7 \times 6 \times \cancel{5!}}{\cancel{5!} \times 4 \times 3 \times 2} \times \frac{x^2 \; (-2^5)^2}{2^4}$$

$$= -9 \times 7 \times 2 \times 2 \; x^2$$

$$= -252x^2.$$

Ex!- $\left(2x - \frac{1}{x}\right)^{10}$ is independent of x.

Sol^n!- let T_{r+1} be independent of x.

$$T_{r+1} = (-1)^r \, {}^{10}C_r \, (2x)^{10-r} \left(\frac{1}{x}\right)^r$$

$(-1)^r \times 10_{C_r} \times (2)^{10-r} \times x^{10-r} \cdot x^{-r}$

$(-1)^r \times 10_{C_r} \times 2^{10-r} \cdot x^{10-2r}$

if binomial theorem of expansion is independent of x.

$x^{10-2r} = x^0$

$10 - 2r = 0$

$2r = 10$

$\boxed{r = 5}$

$\Rightarrow (r+1) = (5+1) = 6$

$T_6 = T_{5+1} = (-1)^5 \times 10_{C_5} \times (2)^{10-5} \times x^0$

$= -1 \times \dfrac{10!}{5! \times 5!} \times 2^5$

$= \dfrac{-1 \times \cancel{10} \times \overset{3}{\cancel{9}} \times \overset{2}{\cancel{8}} \times 7 \times 6 \times \cancel{5!}}{\cancel{5!} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1} \times 2^5$

$= -3 \times 2 \times 7 \times 6 \times 2^5 = -8064.$

Ex:- $(99)^5$.

$(100-1)^5$

$\Rightarrow (100)^5 + 5(100)^4(-1) + 10(100)^3(-1)^2 + 10(100)^2(-1)^3$
$+ 5(100)(-1)^4 + (-1)^5$.

$\Rightarrow (100)^5 - 5(100)^4 + 10(100)^3 - 10(100)^2 + 500 -$

$\Rightarrow 10000000000 - 500000000 + 10000000$
$100000 + 500 - 1$

$\Rightarrow 9509900499$.